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ANALYTICAL SOLUTION OF STOKES FLOW NEAR CORNERS AND APPLICATIONS TO NUMERICAL SOLUTION OF NAVIER-STOKES EQUATIONS WITH HIGH PRECISION

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Abstract

We present analytical solution of the Stokes problem in 2D domains. This is then used to find the asymptotic behavior of the solution in the vicinity of corners, also for Navier-Stokes equations in 2D. We apply this to construct very precise numerical finite element solution.

1. Introduction

The behaviour of the solution of Stokes and Navier-Stokes equations in domains with boundary corners or with discontinuities in boundary conditions is still not quite well understood. The singularities arising in these cases will be analyzed in this paper. Let us note that the asymptotic behaviour applies also to Navier-Stokes equations. In selected cases we use the analytical solution to characterize the singular part of the solution. The results will be applied to two examples: the flow in a channel with forward and backward steps, and the problem of lid driven cavity.

2. Analytical solution of the Stokes flow near corners

2.1. Problem formulation

We consider the Stokes problem for incompressible viscous fluid in two dimensions, in vorticity – stream function formulation, cf. e.g. Feistauer [6],

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} \;=\; -\omega \;, \tag{1}$$

$$\frac{\partial \omega}{\partial t} = \nu\Big(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2}\Big) \ , \tag{2}$$

where $\omega(x,y)$ is the vorticity, $\psi(x,y)$ is the stream function.

To analyze the flow in a domain with corners, we transform the problem to polar coordinates

$$x = r\cos\vartheta \ , \quad y = r\sin\vartheta \ . \tag{3}$$

with the pole in the corner, as e.g. the points P or S on Fig. 1.

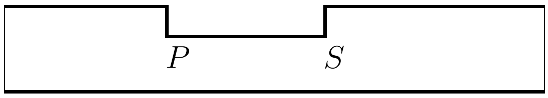

Figure 1: The solution domain Ω.

In the paper we restrict ourselves to the steady flow. Thus the problem (1), (2) in polar coordinates means to find functions $\psi(r,\vartheta)$, $\omega(r,\vartheta)$, satisfying the equations

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \vartheta^2} = -\omega \ , \tag{4}$$

$$\nu\Big(\frac{\partial^2 \omega}{\partial r^2} + \frac{1}{r}\frac{\partial \omega}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \omega}{\partial \vartheta^2}\Big) = 0 \ . \tag{5}$$

Velocity components u_r, u_ϑ are related to the stream function as follows

$$u_r = \frac{1}{r}\frac{\partial \psi}{\partial \vartheta}, \qquad u_\vartheta = -\frac{\partial \psi}{\partial r}. \tag{6}$$

In what follows we also need the equations of motion for Stokes problem in polar coordinates, in velocity - pressure formulation, cf. e.g. Batchelor [1]

$$\nu\left(\frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r}\frac{\partial u_r}{\partial r} + \frac{1}{r^2}\Big(\frac{\partial^2 u_r}{\partial \vartheta^2} - 2\frac{\partial u_\vartheta}{\partial \vartheta} - u_r\Big)\right) - \frac{1}{\rho}\ \frac{\partial p}{\partial r} = 0. \tag{7}$$

$$\nu\left(\frac{\partial^2 u_\vartheta}{\partial r^2} + \frac{1}{r}\frac{\partial u_\vartheta}{\partial r} + \frac{1}{r^2}\Big(\frac{\partial^2 u_\vartheta}{\partial \vartheta^2} + 2\frac{\partial u_r}{\partial \vartheta} - u_\vartheta\Big)\right) - \frac{1}{\rho}\ \frac{1}{r}\ \frac{\partial p}{\partial \vartheta} = 0. \tag{8}$$

Note: without loss of generality, we assume in the paper that the viscosity $\nu = 1$, and also the density $\rho = 1$.

2.2. Analytical solution for singularities

We solve the equations (4), (5) by means of separation of variables, i.e. we seek for the solution in the form

$$\psi(r,\vartheta) = P(r)\cdot F(\vartheta) \ , \tag{9}$$

$$\omega(r,\vartheta) = R(r)\cdot G(\vartheta) \ . \tag{10}$$

Substituting this into (4), (5) we get

$$P''(r)\cdot F(\vartheta)+\frac{1}{r}P'(r)\cdot F(\vartheta)+\frac{1}{r^2}P(r)\cdot F''(\vartheta)=-R(r)\cdot G(\vartheta), \tag{11}$$

$$R''(r)\cdot G(\vartheta)+\frac{1}{r}R'(r)\cdot G(\vartheta)+\frac{1}{r^2}R(r)\cdot G''(\vartheta)=0. \tag{12}$$

Separating the terms with variables r and ϑ in (12) we get

$$\frac{r^2R''(r)+rR'(r)}{R(r)}=-\frac{G''(\vartheta)}{G(\vartheta)}=\varkappa, \tag{13}$$

where $\varkappa$ is a real constant independent of both r and ϑ. This gives two equations:

$$-\varkappa R(r)=0\ ,separR \tag{14}$$

$$G''(\vartheta)+\varkappa\, G(\vartheta)=0. \tag{15}$$

Let us assume $\varkappa>0$. Then the equation (**??**) has the general solution

$$R(r)=a\ r^{-\sqrt{\varkappa}}+b\ r^{\sqrt{\varkappa}}\ , \tag{16}$$

where a, b are arbitrary real constants.

Assumption 1. *As we are interested mainly in the asymptotic behaviour of the solution in the vicinity of corners, in what follows we shall consider only the singular part of the solution*

$$R(r)=a\ r^K. \tag{17}$$

where

$$K=-\sqrt{\varkappa}. \tag{18}$$

Solving the equation(15) for G and using it together with (17) in (10), we get for singular part of the vorticity ω

$$\omega(r,\vartheta)=r^K\Big(c_1\cdot\cos(K\vartheta)+c_2\cdot\sin(K\vartheta)\Big)\quad(+h.o.t), \tag{19}$$

where c_1, c_2 are arbitrary real constants.

Now we substitute this to the equation (11) and get

$$P''(r)F(\vartheta)+\frac{1}{r}P'(r)F(\vartheta)+\frac{1}{r^2}P(r)F''(\vartheta)=-r^K\left(c_1\cos(K\vartheta)+c_2\sin(K\vartheta)\right)\ . \tag{20}$$

From this equation we easily deduce

$$P(r)=r^{K+2}. \tag{21}$$

Using this in (20) we obtain the equation for the function $F(\vartheta)$:

$$F''(\vartheta)+(K+2)^2F(\vartheta)=-c_1\cos(K\vartheta)-c_2\sin(K\vartheta). \tag{22}$$

The general solution of equation (22) is

$$F_{GN}(\vartheta) = D_1 \cos(K+2)\vartheta + D_2 \sin(K+2)\vartheta - \frac{c_1}{4K+4}\cos(K)\vartheta - \frac{c_2}{4K+4}\sin(K)\vartheta, \tag{23}$$

where D_1, D_2, c_1, c_2 are arbitrary real constants. Finally according to (9) and (21) we have for the stream function the asymptotic formula

$$\psi(r,\vartheta) = r^{K+2} \cdot F_{GN}(\vartheta) \quad (+h.o.t.). \tag{24}$$

This result (with still undetermined parameter $K = -\sqrt{\varkappa}$) will be later used for derivation of the asymptotic behaviour of the solution in the vicinity of corners.

3. Singularity of the solution near nonconvex corners

We consider fluid flow in 2D region with boundary corner of internal angle φ, cf. Fig. 1. We assume a rigid boundary and nonslip boundary conditions, so that the boundary conditions for the stream function are

$$\psi(r,0) = 0, \quad \psi(r,\varphi) = 0, \tag{25}$$

$$\frac{\partial\psi}{\partial\vartheta}(r,0) = 0, \quad \frac{\partial\psi}{\partial\vartheta}(r,\varphi) = 0. \tag{26}$$

The stream function $\psi(r,\vartheta)$, according to (23) and (24) is

$$\psi(r,\vartheta) = r^{K+2}\{A_1 \cos(K+2)\vartheta + A_2 \sin(K+2)\vartheta + A_3 \cos K\vartheta + A_4 \sin K\vartheta\}, \tag{27}$$

where

$$A_1 = D_1, \quad A_2 = D_2, \quad A_3 = -\frac{c_1}{4K+4}, \quad A_4 = -\frac{c_2}{4K+4}. \tag{28}$$

The stream function (27) is subject to boundary conditions (25), (26), so we obtain the equations

$$\begin{aligned} A_1 + A_3 &= 0\ , \\ A_1 \cos(K+2)\varphi + A_2 \sin(K+2)\varphi + A_3 \cos(K\varphi) + A_4 \sin(K\varphi) &= 0\ , \\ A_2(K+2) + A_4 K &= 0\ , \\ -A_1(K+2)\sin(K+2)\varphi + A_2(K+2)\cos(K+2)\varphi - & \\ -A_3 K \sin(K\varphi) + A_4 K \cos(K\varphi) &= 0. \end{aligned}$$

In order to determine the parameter K (and consequently $\varkappa$) we have to ensure the condition

$$Q(K) = 0,$$

where

$$Q(K) = \det\begin{pmatrix} 1 & 0 & 1 & 0 \\ \cos(K+2)\varphi & \sin(K+2)\varphi & \cos(K\varphi) & \sin(K\varphi) \\ 0 & K+2 & 0 & K \\ -(K+2)\sin(K+2)\varphi & (K+2)\cos(K+2)\varphi & -K\sin(K\varphi) & K\cos(K\varphi) \end{pmatrix}. \tag{29}$$

We easily get

$$Q(K) = -2(K+2)K + (K^2 + (K+2)^2)\sin(K+2)\varphi\ \sin(K\varphi) + 2K(K+2)\cos(K+2)\varphi\ \cos(K\varphi). \tag{30}$$

This expression will be simplified by means of substitution

$$\gamma = K + 1. \tag{31}$$

Then, after some manipulations we get

$$Q(\gamma) = -4(\gamma^2 \sin^2\varphi - \sin^2\gamma\varphi). \tag{32}$$

So the parameter γ has to satisfy the algebraic equation

$$\gamma^2 \sin^2\varphi - \sin^2\gamma\varphi = 0. \tag{33}$$

Example 1. As an example we take the domain shown in Fig. 1, where the angle

$$\varphi = \frac{3}{2}\pi. \tag{34}$$

Then solving the equation (33) we get

$$\gamma = 0.5444837, \tag{35}$$

so that

$$K = -\sqrt{\varkappa} = \gamma - 1 = -0.45552, \tag{36}$$

Now, following the expression (27) we get for the stream function the asymptotic behaviour near the angle $\frac{3}{2}\pi$:

$$\psi(r,\vartheta) = r^{1.54448} \cdot F(\vartheta), \tag{37}$$

where the function F does not depend on r. Consequently for the velocity components, by (6) we have the asymptotics

$$u_r = r^\gamma\ F_1(\vartheta) = r^{0.54448}\ F_1(\vartheta),$$
$$u_\vartheta = r^\gamma\ F_2(\vartheta) = r^{0.54448}\ F_2(\vartheta), \tag{38}$$

where the functions $F_1(\vartheta)$, $F_2(\vartheta)$ are independent of r.

To derive the asymptotic behaviour for pressure we use the momentum equation (7), where we substitute for u_r and u_ϑ from (38) and get

$$\frac{\partial p}{\partial r} = r^{\gamma-2}\Phi(\vartheta), \tag{39}$$

where the function $\Phi(\vartheta)$ is independent of r. So finally

$$p \approx r^{\gamma-1}\Phi_p(\vartheta) \approx r^{-0.45552}\Phi_p(\vartheta), \tag{40}$$

where the function $\Phi_p(\vartheta)$ is independent of r.

Let us note that the same asymptotics were also found by a different technique in Kondratiev [7], in Ladeveze and Peyret [8] for 2D channel flow, and also in B. [2] in case of the cylindrically symmetric flow. We also note that the asymptotics (38), (40) apply also to Navier-Stokes equations, see e.g. B. [2].

4. Singularity by discontinuous boundary condition

Let us consider 2D flow in lid driven cavity, see Fig. 2, with boundary conditions

$$\psi(r, \frac{3}{2}\pi) = 0, \quad \psi(r, 2\pi) = 0, \tag{41}$$

$$\frac{1}{r}\frac{\partial\psi}{\partial\vartheta}(r, \frac{3}{2}\pi) = 0, \quad \frac{1}{r}\frac{\partial\psi}{\partial\vartheta}(r, 2\pi) = 1, \tag{42}$$

for left upper corner.

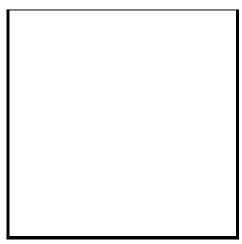

Figure 2: The lid driven cavity.

We solve the equations (1),(2) similarly as we did in Section 2, by means of separation (9) and (10)

$$\psi(r,\vartheta) = P(r)\cdot F(\vartheta),$$
$$\omega(r,\vartheta) = R(r)\cdot G(\vartheta).$$

One can easily derive that it is sufficient to put

$$P(r) = r \tag{43}$$

in order to satisfy the first condition in (42). This immediately implies that the asymptotics of the stream function in upper corners of the cavity are

$$\psi(r,\vartheta) = r\cdot F(\vartheta), \tag{44}$$

where r is the distance from the relevant corner.

Moreover from (11) and (43) it follows

$$\frac{1}{r}F(\vartheta) + \frac{1}{r}F''(\vartheta) = -R(r)\cdot G(\vartheta), \tag{45}$$

and here suitable function $R(r)$ is

$$R(r) = \frac{1}{r}. \tag{46}$$

Then for vorticity we get, using (10)

$$\omega(r,\vartheta) = \frac{1}{r}\cdot G(\vartheta). \tag{47}$$

Further from (46) and (13)

$$\varkappa = 1\ . \tag{48}$$

Now by (15) we have the equation for $G(\vartheta)$:

$$G''(\vartheta) + G(\vartheta) = 0, \tag{49}$$

whose general solution is

$$G(\vartheta) = c_1\cdot\cos\vartheta + c_2\cdot\sin\vartheta, \tag{50}$$

where c_1, c_2 are arbitrary real constants. By (45) we get the equation for $F(\vartheta)$

$$F''(\vartheta) + F(\vartheta) = -c_1\cdot\cos\vartheta - c_2\cdot\sin\vartheta\ . \tag{51}$$

The general solution of (51) is

$$F_{GN}(\vartheta) = A_1\cos\vartheta + A_2\sin\vartheta + \frac{c_2}{2}\vartheta\cos\vartheta - \frac{c_1}{2}\vartheta\sin\vartheta. \tag{52}$$

Then the stream function, using (44) may be written as

$$\psi(r,\vartheta) = r\{A_1\cos\vartheta + A_2\sin\vartheta + A_3\vartheta\cos\vartheta + A_4\vartheta\sin\vartheta\}. \tag{53}$$

The constants $A_1,\ldots,A_4$ are then determined using the boundary conditions (41), (42), and we get the analytical solution for stream function near the corner of the cavity as

$$\psi(r,\vartheta) = rF(\vartheta), \tag{54}$$

where

$$F(\vartheta) = \frac{1}{\frac{\pi^2}{4}-1}\left\{\pi^2\cos\vartheta - \frac{3}{2}\pi\sin\vartheta - \frac{\pi}{2}\vartheta\cos\vartheta + \vartheta\sin\vartheta\right\}. \tag{55}$$

Now by (6) and (44) we get

$$u_r = F'(\vartheta), \quad u_\vartheta = -F(\vartheta), \tag{56}$$

We observe that the velocity components do not depend on the distance r from the cavity corner. Now we put the velocity components (56) to the momentum equation for Stokes problem in polar coordinates (7) and get the expression for pressure

$$\frac{\partial p}{\partial r} = \left(\frac{1}{r^2}(F'''(\vartheta) + F'(\vartheta)) \right) \frac{\nu}{\rho}. \tag{57}$$

For simplicity we assume $\frac{\nu}{\rho} = 1$. Then

$$p(r, \vartheta) = \frac{1}{r}(-F'''(\vartheta) - F'(\vartheta)) + C_1(\vartheta). \tag{58}$$

Using (55) we get

$$p(r, \vartheta) = \frac{1}{r} \frac{1}{\frac{\pi^2}{4} - 1}(\pi \sin\vartheta - 2\cos\vartheta) + C_1(\vartheta)\ , \vartheta \in \left(\frac{3}{2}\pi, 2\pi \right). \tag{59}$$

So we obtained the asymptotic expression for pressure with respect to r coordinate. Let us note that the asymptotic expression $p(r, \vartheta) = \frac{1}{r}\Phi(\vartheta)$ was found already by Luchini [9]. We followed some of his ideas in this section.

5. Application to finite element solution of Navier-Stokes equations

In this section we deal with isothermal flow of Newtonian viscous fluids with constant density. The flow is modelled by the Navier-Stokes system of partial differential equations (nonconservative form). We deal only with steady flow:

$$(\mathbf{u} \cdot \nabla)\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega, \tag{60}$$

$$\nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega, \tag{61}$$

where

- $\mathbf{u} = (u_1, u_2)^T$ means the vector of flow velocity, in m/s, being a function of $\mathbf{x}$,
- $p = \frac{p_r}{\rho}$ is the pressure divided by the density considered in Pa m^2/kg,
- $\nu = \frac{\mu}{\rho}$ denotes the kinematic viscosity of the fluid considered in m^2/s,
- $\mathbf{f}$ denotes the density of volume forces per mass unit considered in N/m^2.

The system is supplied with the boundary conditions

$$\mathbf{u} = \mathbf{g} \quad \text{on } \Gamma. \tag{62}$$

Here $\mathbf{g}$ is a given function of $\mathbf{x}$ satisfying $\int_\Gamma \mathbf{g} \cdot \mathbf{n}\ d\Gamma = 0$, where $\mathbf{n}$ denotes the unit outer normal vector to the boundary Γ.

5.1. Finite element solution: a priori error estimates

For the approximate solution of the Navier-Stokes equation we use the finite element method with Taylor-Hood elements. In the paper we utilize the a priori estimate of the finite element error for the Navier-Stokes equations (60)–(61) (cf. [5])

$$\|\nabla(\mathbf{u}-\mathbf{u_h})\|_{L_2(\Omega)} \le C\Big[\Big(\sum_K h_K^{2k} \mid \mathbf{u} \mid^2_{H^{k+1}(T_K)}\Big)^{1/2} + \Big(\sum_K h_K^{2k} \mid p \mid^2_{H^k(T_K)}\Big)^{1/2}\Big], \quad (63)$$

$$\|p-p_h\|_{L_2(\Omega)} \le C\Big[\Big(\sum_K h_K^{2k} \mid \mathbf{u} \mid^2_{H^{k+1}(T_K)}\Big)^{1/2} + \Big(\sum_K h_K^{2k} \mid p \mid^2_{H^k(T_K)}\Big)^{1/2}\Big], \quad (64)$$

where $\mathbf{u}$, p are in turn the precise velocity vector and precise pressure, and $\mathbf{u_h}$, p_h are in turn the approximate velocity vector and approximate pressure, h_K is the diameter of triangle T_K of a triangulation $\mathcal{T}$, and $k = 2$ for Taylor-Hood elements.

Remark: In [2] we have shown that the asymptotic behaviour of the solution near corners derived for the Stokes flow applies also to Navier Stokes equations. We also suggested an algorithm for generation of the finite element mesh near corners that makes use of the information on the asymptotic behaviour of the solution of Navier-Stokes equations.

5.2. Algorithm for generation of computational mesh

Now we combine the results of Subsection 5.1 and Section 4. By (58), the leading term of expansion for pressure is

$$p(r,\vartheta) = r^{-1}\Phi(\vartheta) + \dots, \quad (65)$$

where r is the distance from the corner, ϑ the angle and Φ is a smooth function. Taking the expansion (65), we can estimate the seminorm of p:

$$\mid p \mid^2_{H^k(T_K)} \approx C \int_{r_K-h_K}^{r_K} \rho^{2(-k-1)}\ \rho\, d\rho = C\left[-r_K^{-2k} + (r_K-h_K)^{-2k}\right] \quad (66)$$

where r_K is the distance of element T_K from the corner.

Putting estimate (66) into the a priori estimate (63) or (64), we derive that we should guarantee

$$h_K^{2k}\left[-r_K^{-2k} + (r_K-h_K)^{-2k}\right] \approx h_{ref}^{2k} \quad (67)$$

in order to get the error estimate of order $O(h_{ref}^k)$ uniformly distributed on elements.

From this expression, we compute element diameters in accordance to chosen h_{ref}.

For evaluating the achieved accuracy of the approximate solution, we use the a posteriori error estimator, see e.g. [3].

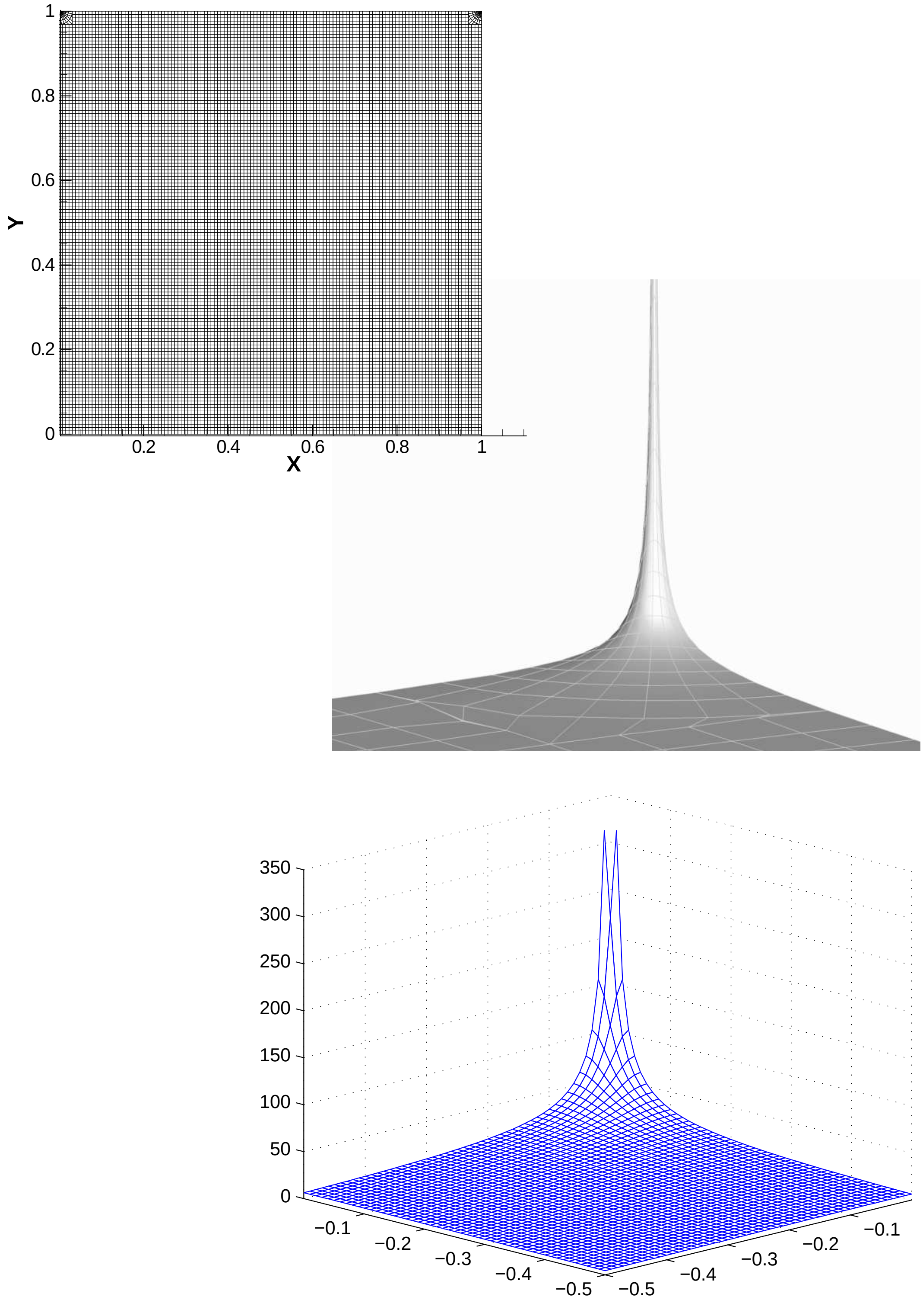

Figure 3: Lid driven cavity. Top: mesh 128×128 refined locally near upper corners. Centre: pressure near left upper corner by adjusted finite elements, Re = 10,000. Bottom: pressure near left upper corner analytically.

5.3. Numerical results

We show on Fig. 3 some results for lid driven cavity. First is the locally refined mesh near upper corners of the cavity, obtained by the algorithm described in Section 5.2. This mesh is then used for very precise finite element solution: on central part of Fig. 3 we show the pressure calculated from the Navier-Stokes equations on this mesh. For comparison, we also give the graph of pressure obtained by the analytical solution (59) of the Stokes flow.

Concerning applications to flow in 2D channel like that on Fig. 1, we refer to [4], where there are also tables showing the high precision of solution on such meshes.

6. Conclusion

In the paper we are interested in Stokes and Navier-Stokes problem with singularities caused either by nonconvex corners in 2D domains or by discontinuities in boundary conditions. For the Stokes flow we find analytically the principal part of the asymptotics of solution in the vicinity of corners. This result is used on one hand to construct the finite element mesh adjusted to singularity. This mesh is then used to find very precise solution of Navier-Stokes equations. On the other hand, the analytical solution of the Stokes flow near corners of lid driven cavity, e.g., may be used to test other methods.

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