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IDENTIFICATION OF PARAMETERS IN PARABOLIC INVERSE PROBLEMS

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Abstract

In this paper we consider a parabolic inverse problem in which two unknown functions are involved in the boundary conditions, and attempt to recover these functions by measuring the values of the flux on the boundary. Explicit solutions for the temperature and the radiation terms are derived, and some stability and asymptotic results are discussed. Finally, by using the newly proposed numerical procedure some computational results are presented.

1. Introduction

It is well known that the heat temperature is a function of radiative heat flux. In certain heat transfer it is of interest to devise methods to evaluate temperature functions by using only measurable radiation taken outside the medium.

This paper seeks to determine some unknown temperature functions which depend only on the heat flux in a heat transfer equation.

The problem of determining unknown parameters in parabolic differential equations has been treated previously by many authors [1, 2, 3, 4, 9, 12, 14, 15, 16, 18]. Usually these problems involve the determination of a single unknown parameter from additional boundary data. In some applications, however, it is desirable to be able to determine more than one parameter from the given boundary data [5].

Hence, we may consider the following problem:

$$u_t = u_{xx}, \quad 0 < x < 1, \quad 0 < t < T, \tag{1}$$

$$u(x,0) = f(x), \qquad 0 < x < 1, \tag{2}$$

$$u(0,t) = G(u_x(0,t)) + g_0(t), \qquad 0 < t < T, \tag{3}$$

$$u(1,t) = H(u_x(1,t)) + h_0(t), \qquad 0 < t < T, \tag{4}$$

with the additional conditions:

$$u_x(0,t) = g_1(t), \qquad 0 < t < T, \tag{5}$$

$$u_x(1,t) = h_1(t), \qquad 0 < t < T, \tag{6}$$

where T is a given constant, $f(x)$, $g_0(t)$, $h_0(t)$, $g_1(t)$, and $h_1(t)$ are given functions. The equation (1) may be used to describe the flow of heat in a rod. Hence, we might think of this problem as the problem of determining the unknown temperature terms of rod. In this context, $g_0(t)$ and $h_0(t)$ are known functions which depend on the temperature at the positions $x = 0$ and $x = 1$.

If the functions G and H are given, there may be no solution for the problem (1)–(6). On the other hand, when G and H are known, under certain conditions there may exist a unique solution for the problem (1)–(4), and this solution may not satisfy the additional conditions (5) and (6). In this case, we say that the pair of functions $(u, (G, H))$ provides a solution to the inverse problem (1)–(6). It is well known that the inverse problem (1)–(6) has a unique solution and also some more applications have been discussed in [10, 7, 8, 6].

The outline of this paper is as follows: In 2, some representation results are established. In 3 and 4, some monotonic, stability and asymptotic behavior results of solutions are discussed. In 5, by using the theta function, we consider a priori estimates of solutions. A numerical scheme is described in 6. In the final section we compare the solutions of the problem (1)–(6) obtained by theta function and by other numerical methods, respectively.

2. Representation formula

To solve the inverse problem (1)–(6), let us consider the following auxiliary problem:

$$u_t = u_{xx}, \qquad 0 < x < 1, \quad 0 < t < T, \tag{7}$$

$$u(x,0) = f(x), \qquad 0 \le x \le 1, \tag{8}$$

$$u_x(0,t) = g_1(t), \qquad t \ge 0, \tag{9}$$

$$u_x(1,t) = h_1(t), \qquad t \ge 0. \tag{10}$$

For any piecewise-continuous functions f, g_1, and h_1, this problem has a unique solution [2] as follows:

$$\begin{aligned} u(x,t) = & \int_0^1 \{\theta(x-\xi,t) + \theta(x+\xi,t)\} f(\xi) d\xi \\ & - 2\int_0^t \theta(x, t-\tau) g_1(\tau) d\tau + 2\int_0^t \theta(x-1, t-\tau) h_1(\tau) d\tau, \end{aligned} \tag{11}$$

where

$$\theta(x,t) = \sum_{m=-\infty}^{\infty} K(x+2m, t),$$

and

$$K(x,t) = \frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{x^2}{4t}\right), \quad t > 0.$$

(3) and (5) yield

$$G(g_1(t)) = u(0,t) - g_0(t). \tag{12}$$

From $\theta(-\xi,t) = \theta(\xi,t)$, (11) and (12) we know that

$$\begin{aligned} G(g_1(t)) = &2\int_0^1 \theta(\xi,t)f(\xi)d\xi \\ &- 2\int_0^t \theta(0,t-\tau)g_1(\tau)d\tau + 2\int_0^t \theta(-1,t-\tau)h_1(\tau)d\tau - g_0(t). \end{aligned}$$

If we assume that the function $s = g_1(t)$ is invertible, then we find

$$\begin{aligned} G(s) = &2\int_0^1 \theta(\xi, g_1^{-1}(s))f(\xi)d\xi \\ &- 2\int_0^{g_1^{-1}(s)} \theta(0, g_1^{-1}(s)-\tau)g_1(\tau)d\tau \\ &+ 2\int_0^{g_1^{-1}(s)} \theta(-1, g_1^{-1}(s)-\tau)h_1(\tau)d\tau - g_0(g_1^{-1}(s)). \end{aligned} \tag{13}$$

Similarly, for H we have

$$\begin{aligned} H(\nu) = &2\int_0^1 \theta(\xi+1, h_1^{-1}(\nu))f(\xi)d\xi \\ &- 2\int_0^{h_1^{-1}(\nu)} \theta(1, h_1^{-1}(\nu)-\tau)g_1(\tau)d\tau \\ &+ 2\int_0^{h_1^{-1}(\nu)} \theta(0, h_1^{-1}(\nu)-\tau)h_1(\tau)d\tau - h_0(h_1^{-1}(\nu)), \end{aligned} \tag{14}$$

where the invertible function $\nu = h_1(t)$ may be obtained from $\theta(1-\xi,t) = \theta(1+\xi,t)$ and conditions (4) and (6).

3. Some monotonic results

In this section, we consider some monotonic results. First, by demonstrating the following statement, we discuss the strict monotony of solutions.

Theorem 3.1. *If g_0 and $g_1 = h_1$ are strictly decreasing and continuous functions and $f = 0$, then G is a strictly decreasing function.*

Proof. By differentiating (11) with respect to t, we obtain

$$\begin{aligned} u_t(x,t) =& w_t(x,t) - 2\left\{\theta(x,0)g_1(t) + \int_0^t \frac{\partial^2\theta}{\partial x^2}(x,t-\tau)g_1(\tau)d\tau\right\} \\ &+ 2\left\{\theta(x-1,0)h_1(t) + \int_0^t \frac{\partial^2\theta}{\partial x^2}(x-1,t-\tau)h_1(\tau)d\tau\right\}, \end{aligned}$$

where

$$w(x,t) = \int_0^1 \{\theta(x-\xi,t) + \theta(x+\xi,t)\}f(\xi)d\xi.$$

It follows from the properties of $\theta(x,t)$ function

$$\lim_{\tau\uparrow t}\theta(x,t-\tau) = 0, \qquad 0 < x < 1,$$

$$\begin{aligned} u_t(0,t) =& -2\int_0^t \frac{\partial^2\theta}{\partial x^2}(0,t-\tau)g_1(\tau)d\tau \\ &+ 2\int_0^t \frac{\partial^2\theta}{\partial x^2}(-1,t-\tau)h_1(\tau)d\tau, \end{aligned}$$

$$\frac{\partial^2\theta}{\partial x^2} = \frac{\partial\theta}{\partial t} = -\frac{\partial\theta}{\partial\tau},$$

and integration by parts that

$$\begin{aligned} u_t(0,t) =& -2\theta(0,t)g_1(0) - 2\int_0^t \theta(0,t-\tau)g_1'(\tau)d\tau \\ &+ 2\theta(-1,t)h_1(1) + 2\int_0^t \theta(-1,t-\tau)h_1'(\tau)d\tau. \end{aligned}$$

From $g_1(0) = h_1(0) = 0$, $2\theta(0,t) > 1$ and $0 < 2\theta(-1,t) < 1$, we conclude that

$$-\beta(g_1(t) - g_1(0)) + (h_1(t) - h_1(0)) = g_1(t)(1-\beta) > 0, \tag{15}$$

where

$$\beta = \sup_{0\le t\le T}\{2\theta(0,t)\} > 1,$$

and $h_1 = g_1$ is a strictly decreasing function. From (15), we obtain $u_t(0,t) > 0$. Now from (12) and $s = g_1(t)$, we find

$$G'(s) = \left(\frac{\partial u(0,t)}{\partial t} - \frac{\partial g_0(t)}{\partial t}\right)\frac{1}{g_1'(t)} < 0.$$

This demonstrates that G is a strictly decreasing function. □

When h_0 and $g_1 = h_1$ are strictly decreasing and continuous functions and $f = 0$, we can obtain a similar result for H.

4. Stability and asymptotic results

In this section, we consider the stability of the solutions G and H, and have the following statement.

Theorem 4.1. *Let* $(u_i, (G_i, H_i))$ $(i = 1, 2)$ *be two solutions of the problem (1)–(6) corresponding to the two given data* $f = 0$, $g_{11}(0)=g_{12}(0)=0$ *and* $h_{11}(0)=h_{12}(0)=0$. *Then, these solutions are stable.*

Proof. From (11) we have

$$u(x,t) = -2\int_0^t \theta(0,t-\tau)g_1(\tau)d\tau + 2\int_0^t \theta(-1,t-\tau)h_1(\tau)d\tau.$$

Hence, by (12) we find

$$\begin{aligned} G_1 - G_2 = &- (g_{01}(t) - g_{02}(t)) \\ &- 2\left(\int_0^t \theta(0,t-\tau)g_{11}(\tau)d\tau - \int_0^t \theta(-1,t-\tau)g_{12}(\tau)d\tau\right) \\ &+ 2\left(\int_0^t \theta(-1,t-\tau)h_{11}(\tau)d\tau - \int_0^t \theta(-1,t-\tau)h_{12}(\tau)d\tau\right) \\ = &- (g_{01}(t) - g_{02}(t)) \\ &- 2\int_0^t \theta(0,t-\tau)\{g_{11}(\tau) - g_{12}(\tau)\}d\tau \\ &+ 2\int_0^t \theta(-1,t-\tau)\{h_{11}(\tau) - h_{12}(\tau)\}d\tau, \end{aligned}$$

where

$$\begin{aligned} |G_1 - G_2| \le |g_{01} - g_{02}| &+ \left|\int_0^t 2\theta(0,t-\tau)(g_{11} - g_{12})d\tau\right| \\ &+ \left|\int_0^t 2\theta(-1,t-\tau)(h_{11} - h_{12})d\tau\right|, \end{aligned} \tag{16}$$

and from $2\theta(0,t) > 1$ we obtain

$$\begin{aligned} \left|\int_0^t 2\theta(0,t-\tau)(g_{11} - g_{12})d\tau\right| &\le \sup_{0\le t\le T}\{2\theta(0,t)\}\int_0^t |g_{11} - g_{12}|d\tau \\ &\le 2\beta T|g_{11} - g_{12}|, \end{aligned} \tag{17}$$

since integration by parts yields

$$\begin{aligned} \left|\int_0^t (g_{11}(\tau) - g_{12}(\tau))d\tau\right| &\le t|g_{11}(t) - g_{12}(t)| + \left|\int_0^t \tau(g'_{11}(\tau) - g'_{12}(\tau))d\tau\right| \\ &\le 2T|g_{11}(t) - g_{12}(t)|, \end{aligned}$$

where

$$\beta = 2 \sup_{0 \le t \le T} \theta(0, t).$$

In this manner, we can also prove

$$\left| \int_0^t 2\theta(-1, t-\tau)(h_{11} - h_{12}) d\tau \right| \le 2T|h_{11} - h_{12}|. \tag{18}$$

Hence, from (17), (18), and (16) we obtain

$$|G_1 - G_2| \le |g_{01} - g_{02}| + 2\beta T|g_{11} - g_{12}| + 2T|h_{11} - h_{12}|.$$

Similar results may be obtained for the solution H. Now, the stability of u can be easily proved by using the equation (1) and conditions (2), (5), and (6). This completes the proof of the theorem. □

In the remainder of this section, by giving the following statement, we prove an asymptotic boundary behavior result of the solution to the problem (1)–(6), which agrees with the physical experiments of radiative heat transfer.

Theorem 4.2. *If $f = g_1 = h_1 = 0$, G, and H are either increasing or decreasing functions for any given functions $g_0'(t) < 0$ and $h_0'(t) < 0$ or $g_0'(t) > 0$ and $h_0'(t) > 0$, respectively, then we have*

$$G = -g_0(t), \qquad H = -h_0(t).$$

Proof. From (13) and (14) we obtain

$$\frac{dG}{dt} = -g_0'(t), \qquad \frac{dH}{dt} = -h_0'(t).$$

Now if $g_0'(t) > 0$ and $h_0'(t) > 0$, then $\frac{dG}{dt} < 0$ and $\frac{dH}{dt} < 0$, i.e., G and H are decreasing functions. Similarly, G and H are increasing functions in the case that $g_0'(t) < 0$ and $h_0'(t) < 0$.

The final part of the above statement can be easily proved by using (13) and (14), and thus we conclude that

$$G = -g_0(t), \qquad H = -h_0(t).$$

□

The above result agrees with the law of radiation of a solid, like Newton's law of cooling and the Stefan's law of radiation.

5. A priori estimates of solutions

In this section, we consider a priori estimates of the solutions G and H using the a-prior estimate of $\theta(x,t)$. From [2] we obtain

$$\theta(0,t) = \frac{1}{\sqrt{4\pi t}}\left(1 + 2\sum_{m=1}^{\infty}\exp(\frac{-m^2}{t})\right),$$

and

$$\exp\left(\frac{-m^2}{t}\right) < \frac{t}{m^2}.$$

Therefore, we have

$$\theta(0,t) < \frac{1}{\sqrt{\pi t}}\left(1 + \frac{\pi^2}{3}t\right). \tag{19}$$

It follows from

$$\theta(-1,t) = \frac{1}{\sqrt{4\pi t}}\left(1 + 2\sum_{m=1}^{\infty}\exp(-\frac{(2m-1)^2}{4t})\right),$$

and

$$\exp(-x) < \frac{1}{x}$$

that

$$\theta(-1,t) < \frac{1}{\sqrt{4\pi t}}\left(1 + \frac{\pi^2}{2}t\right). \tag{20}$$

Hence, from (19), (20), and (13) we know

$$\begin{aligned}\tilde{G}(s) = & -200\int_0^{\frac{s}{100}} \frac{1}{\sqrt{\pi(\frac{s}{100}-\tau)}}\left(1 + \frac{\pi^2}{3}(\frac{s}{100}-\tau)\right)\tau d\tau \\ & + 10\int_0^{\frac{s}{100}} \frac{1}{\sqrt{\pi(\frac{s}{100}-\tau)}}\left(1 + \frac{\pi^2}{2}(\frac{s}{100}-\tau)\right)\tau d\tau. \end{aligned} \tag{21}$$

Now, by computing (21) we obtain

$$\tilde{G}(s) = \frac{s^{\frac{3}{2}}\,(1000 + \pi^2\,s)}{75000\,\sqrt{\pi}} - \frac{s^{\frac{3}{2}}\,(1500 + \pi^2\,s)}{11250\,\sqrt{\pi}},$$

where $\tilde{G}(s)$ is an a priori estimate for $G(s)$.

6. The finite difference scheme

The domain $[0,1] \times [0,T]$ is divided into an $M \times N$ mesh with the spatial step size $h = 1/M$ in the x direction and the time step size $\tau = T/N$, respectively.

Grid points (x_i, t_n) are defined by

$$x_i = ih, \qquad i = 0,1,2,\cdots,M,$$

$$t_n = n\tau, \qquad n = 0,1,2,\cdots,N,$$

in which M and N are integers. The notation u_i^n stands for the finite difference approximation to $u(ih, n\tau)$.

We consider the use of a weighted average of the centered–difference approximation to u_{xx} at time levels n and $n+1$ in the equation (1) approximated at the point $(ih, (n+r)\tau)$, $0 \le r \le 1$, namely,

$$u_t|_i^{(n+r)} = u_{xx}|_i^{(n+r)} .$$

The space derivative can be written as a weighted average of the values of time levels n and $n+1$. Writing the space derivatives in centered–difference form then yields:

$$u_{xx} \approx r\frac{u_{i+1}^{n+1} + u_{i-1}^{n+1} - 2u_i^{n+1}}{h^2} + (1-r)\frac{u_{i+1}^n + u_{i-1}^n - 2u_i^n}{h^2}.$$

Therefore, we obtain

$$\frac{u_i^{n+1} - u_i^n}{\tau} = r\frac{u_{i+1}^{n+1} + u_{i-1}^{n+1} - 2u_i^{n+1}}{h^2} + (1-r)\frac{u_{i+1}^n + u_{i-1}^n - 2u_i^n}{h^2}. \qquad (22)$$

The unknown values of u at the $(n+1)$th time level may be expressed in terms of the known values of u at the nth time level by writing (22) in the following form:

$$-rsu_{i-1}^{n+1} + (1+2rs)u_i^{n+1} - rsu_{i+1}^{n+1} = s(1-r)u_{i-1}^n + [1-2s(1-r)]u_i^n + s(1-r)u_{i+1}^n, \quad (23)$$

for $i = 1, \cdots, M-1$, where $s = \tau/h^2$.

The resulting system of equations may be used to obtain approximate solutions for the one-dimensional heat equation with two unknown boundary conditions if it is stable in the process of stepping in time and if the resulting system of equations can be solved at each time level.

The application of the von Neumann stability analysis leads to the time weighted scheme based on formula (23) [13], which is conditionally stable and requires

$$0 < s \le \frac{1}{2(1-2r)} \qquad \text{if} \qquad 0 \le r < \frac{1}{2},$$

$$s > 0 \qquad \text{if} \qquad \frac{1}{2} \le r \le 1.$$

The modified equivalent partial differential equation of this method is in the following form [1, 17]:

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} - \frac{h^2}{12}[1 - 6s(1-2r)]\frac{\partial^4 u}{\partial x^4} - \frac{h^4}{360}[1 - 30s(1-2r) + 120s^2(1-3r+3r^2)]$$
$$\frac{\partial^6 u}{\partial x^6} + \mathrm{O}(h^6) = 0.$$

Thus, the finite difference formula (23) is consistent with the one-dimensional heat equation with a truncation error which is generally $\mathrm{O}(h^2)$.

Therefore, when

$$r = \frac{1}{2} - \frac{1}{12s},$$

the term of $\mathrm{O}(h^2)$ disappears and the formula (23) is fourth-order accurate.

Setting $r = 1/2$ in the weighted formula (23) yields

$$-su_{i-1}^{n+1} + 2(1+s)u_i^{n+1} - su_{i+1}^{n+1} = su_{i-1}^n + 2(1-s)u_i^n + su_{i+1}^n,$$

for $i = 1, \cdots, M-1$.

This formula is known as the Crank-Nicolson method, which is unconditionally von Neumann stable for all $s > 0$. Alternatively, time stepping stability criteria for the Crank-Nicolson method can be found by using the matrix technique [11].

7. Numerical results

In this section, we compare the solutions (13) and (14) of the problem (1)–(6) with respect to theta function with some experimental results.

Example 7.1. *We consider (1)–(6), and apply the Crank–Nicolson method. For this purpose, we choose* $f = g_0 = h_0 = 0$, $g_1(t) = 100t$, $h_1(t) = 5t$, $\delta t = 0.0025$, *and* $\delta x = 0.05$. *For the calculation of* $\theta(x,t)$, *we use the first 51 terms of its series.*

Then (13) can be written as follows:

$$G(s) = \frac{-2s^{\frac{3}{2}}}{15\sqrt{\pi}} + \frac{5}{\pi}\left(\frac{-\left(4\sqrt{\pi} - 3\Gamma(-\frac{3}{2}, 0, \frac{25}{s})\right)}{24} - \frac{s\left(2\sqrt{\pi} + \Gamma(t - \frac{1}{2}, 0, \frac{25}{s})\right)}{200}\right)$$

$$
\begin{aligned}
&+\frac{5}{\pi}\left(\sum_{m=1}^{50}\left\{\frac{-\,(\sqrt{\pi}|-1+2\,m|)}{6}\right.\right.\\
&+\frac{2\,m\,\sqrt{\pi}\,|-1+2\,m|}{3}-\frac{2\,m^2\,\sqrt{\pi}\,|-1+2\,m|}{3}\\
&+\frac{20\,m^2\,|m|\,\left(4\,\sqrt{\pi}-3\,\Gamma(-\frac{3}{2},0,\frac{100\,m^2}{s})\right)}{3}\\
&+\frac{|-1+2\,m|\,\Gamma\left(-\frac{3}{2},0,\frac{-25\left(-1+4\,m-4\,m^2\right)}{s}\right)}{8}\\
&-\frac{m\,|-1+2\,m|\,\Gamma\left(-\frac{3}{2},0,\frac{-25\left(-1+4\,m-4\,m^2\right)}{s}\right)}{2}\\
&+\frac{m^2\,|-1+2\,m|\,\Gamma\left(-\frac{3}{2},0,\frac{-25\left(-1+4\,m-4\,m^2\right)}{s}\right)}{2}\\
&+\frac{s\,|m|\,\left(2\,\sqrt{\pi}+\Gamma\left(-\frac{1}{2},0,\frac{100\,m^2}{s}\right)\right)}{5}\\
&\left.\left.+\frac{s\left(-\,(\sqrt{\pi}|-1+2m|)-\frac{|-1+2m|}{2}\Gamma\left(-\frac{1}{2},0,\frac{-25\left(-1+4m-4m^2\right)}{s}\right)\right)}{100}\right\}\right).
\end{aligned}
$$

The result for G is plotted in Figure 1. After four hundred time steps, we observe that in Figure 1(b) both the numerical method and the θ–function method are of the second order accuracy.

Example 7.2. *In the problem (1)–(6), let*

$$
\begin{aligned}
f(x) &= x^4,\\
g_1(t) &= 0,\\
h_1(t) &= 2+24t,
\end{aligned}
$$

and

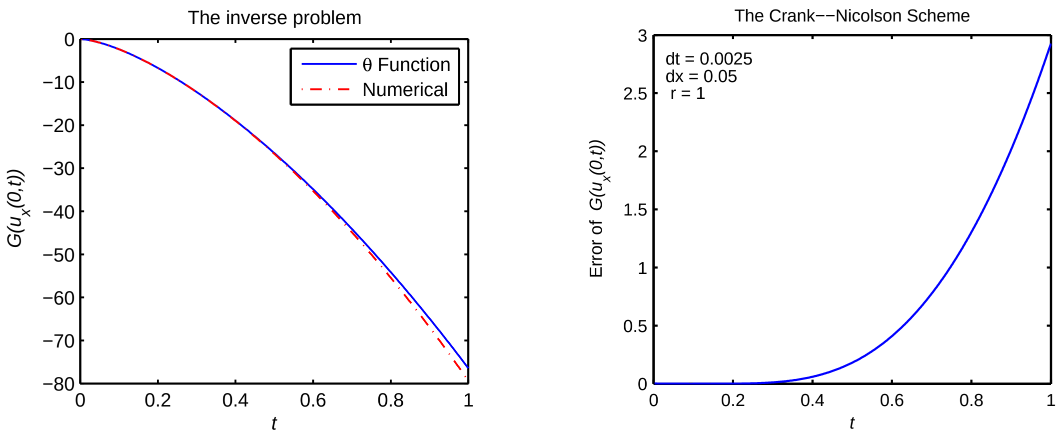
$$g_0(t) = h_0(t) = 0.$$

It is easy to check that the exact solutions $u(x,t)$, $G(u_x(0,t))$ and $H(u_x(1,t))$ are as follows:

$$u(x,t) = 12tx^2 + x^4 + 12t^2,$$

and

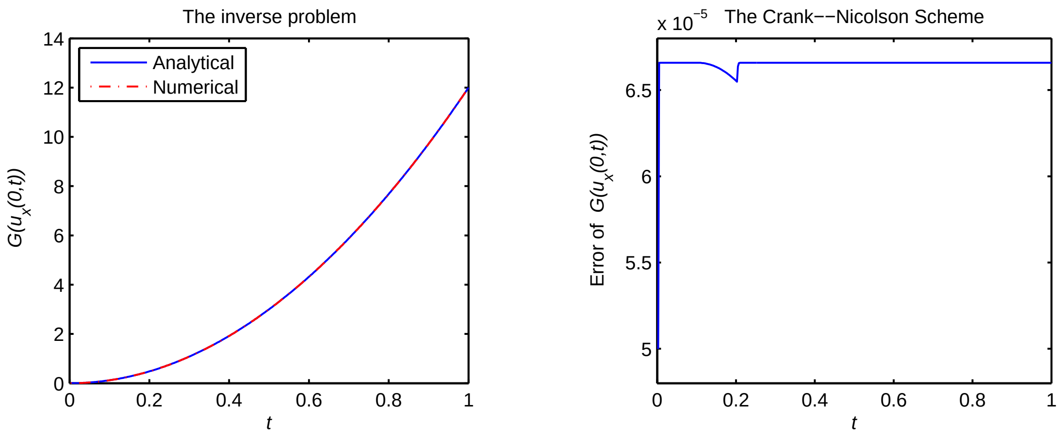
$$
\begin{aligned}
G(u_x(0,t)) &= 12t^2,\\
H(u_x(1,t)) &= 1+12t+12t^2.
\end{aligned}
$$



(a) The θ function and the numerical solutions of $G(s)$.

(b) The error distribution of $G(s)$.

Figure 1: The Crank–Nicolson scheme and the θ function scheme for $G(s)$.

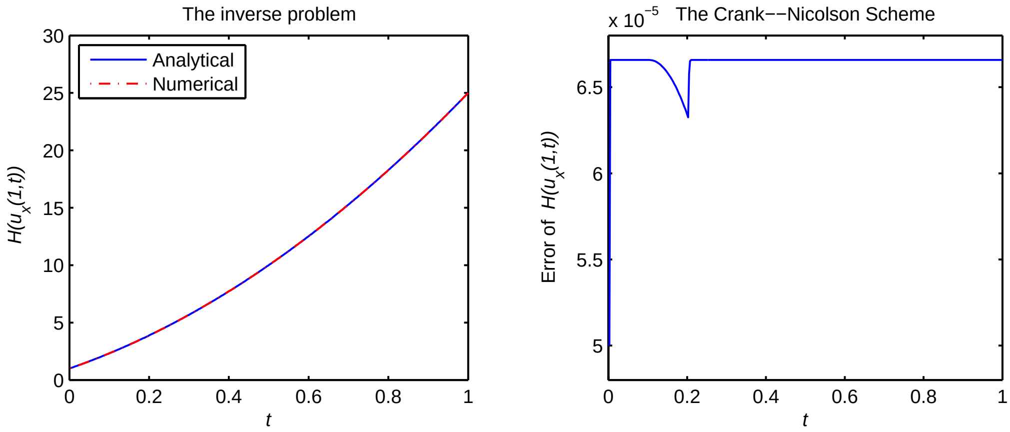


(a) The analytical and the numerical solutions of $G(u_x(0,t))$.

(b) The error distribution of $G(u_x(0,t))$.

Figure 2: The Crank–Nicolson scheme for determining $G(s)$.

We plot $G(u_x(0,t))$ from 0 to 1 in Figure 2(a) and the error distribution of $G(u_x(0,t))$ in Figure 2(b). We also do this for $H(u_x(1,t))$ in Figure 3. It can be observed that the numerical and the analytical results overlap each other. This is the best we can expect from the scheme and the formulas we use.



(a) The analytical and the numerical solutions of $H(u_x(1,t))$.

(b) The error distribution of $H(u_x(1,t))$.

Figure 3: The Crank–Nicolson scheme for determining $H(s)$.

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