A curious property of oscillatory FEM solutions of one-dimensional convection-diffusion problems

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AM2012 (Michal Křížek 60)

3 May 2012

Joint work with

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Background paper SYZ07

Song, Q. S., Yin, G., and Zhang, Z.:
An ε-uniform finite element method for singularly perturbed two-point boundary value problems.
Int. J. Numer. Anal. Model. **4** (2007), 127–140.

The convection-diffusion problem

Two-point boundary value problem

$$-\varepsilon u'' + au' + bu = f \quad \text{on } (0,1), \qquad u(0) = u(1) = 0,$$

where the parameter ε satisfies $0 < \varepsilon \ll 1$,
while $a, b, f \in C[0,1]$ with $a > 0$ and $b \geq 0$.

Problems such as this, where convection dominates diffusion, typically have solutions that are well-behaved away from $x = 1$ but near $x = 1$ change rapidly.
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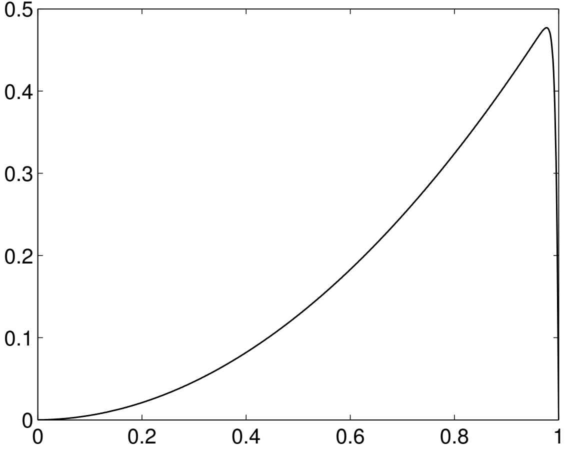
Our generic example

All figures are for the particular example

$$-\varepsilon u'' + u' = x \quad \text{on } (0,1), \qquad u(0) = u(1) = 0,$$

with $\varepsilon = 5 \times 10^{-3}$.



Weak form, trial and test spaces V_h

Weak form of problem: find $u \in H_0^1(0,1)$ satisfying

$$\int_0^1 [\varepsilon u'(x)v'(x) + a(x)u'(x)v(x) + b(x)u(x)v(x)]\,dx = \int_0^1 f(x)v(x)\,dx \quad \forall v \in H_0^1(0,1).$$

Mesh is $0 = x_0 < x_1 < x_2 < \cdots < x_N = 1$.

Assume $N \ll \varepsilon^{-1}$.

For $i = 1, 2, \dots, N-1$, let $\phi_i \in C[0,1]$ be the standard finite element piecewise linear function that satisfies $\phi_i(x_j) = \delta_{ij}$ and support $\phi_i = [x_{i-1}, x_{i+1}]$.

Set $V_h = \text{span}\,\{\phi_1, \phi_2, \dots, \phi_{N-1}\}$, so $V_h \subset H_0^1(0,1)$.

Piecewise linear Galerkin finite element method

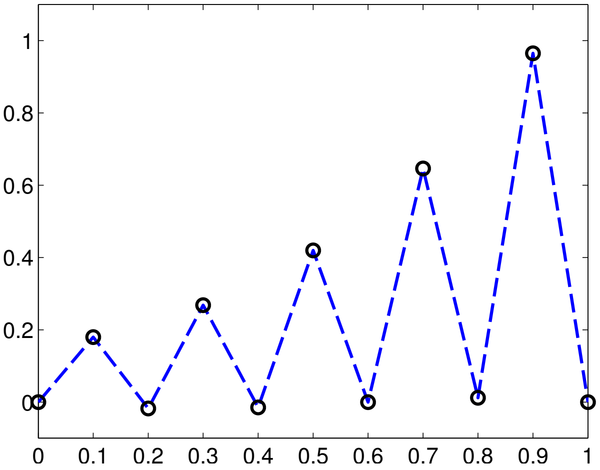
P.w.linear Galerkin finite element solution $u_h \in V_h$ defined by:

$$\int_0^1 [\varepsilon u_h'(x)\phi_i'(x) + a_i u_h'(x)\phi_i(x) + b_i u_h(x)\phi_i(x)]\,dx = \int_0^1 f(x)\phi_i(x)\,dx \quad \text{for } i = 1, 2, \ldots, N-1.$$

Nonstandard quadrature rule: $a(x)$ and $b(x)$ replaced by constants $a_i := a(x_i)$ and $b_i := b(x_i)$ associated with test function ϕ_i in order to generate finite difference scheme having a certain structure.

This is the only numerical method considered in this talk; we shall apply it to various problems.

Computed solution on a uniform mesh with $N \ll \varepsilon^{-1}$

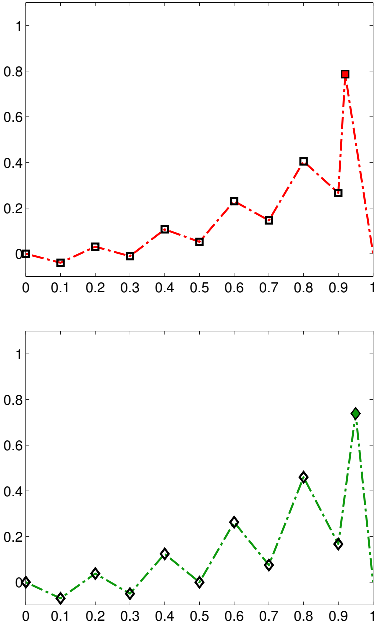


Adding a point to the mesh

SYZ07 modify the uniform mesh in the Galerkin method by adding a mesh point (arbitrarily chosen) to the mesh interval $(0.9, 1)$ where the boundary layer lies.

Next two figures show the computed solutions when the additional mesh points are 0.92 and 0.95 respectively.

Computed solution with additional mesh points $0.92, 0.95$



SYZ07 idea: superimpose these three computed solutions

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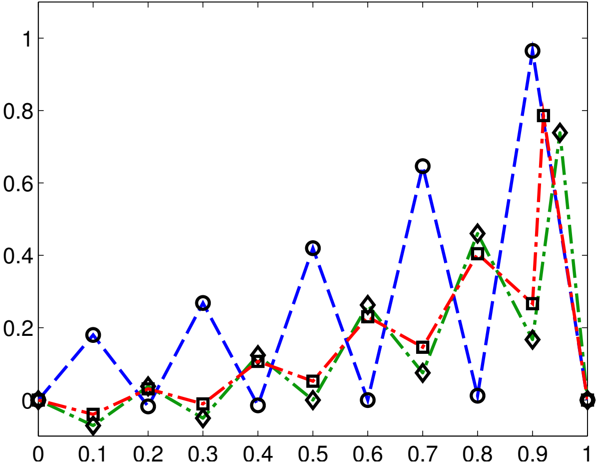
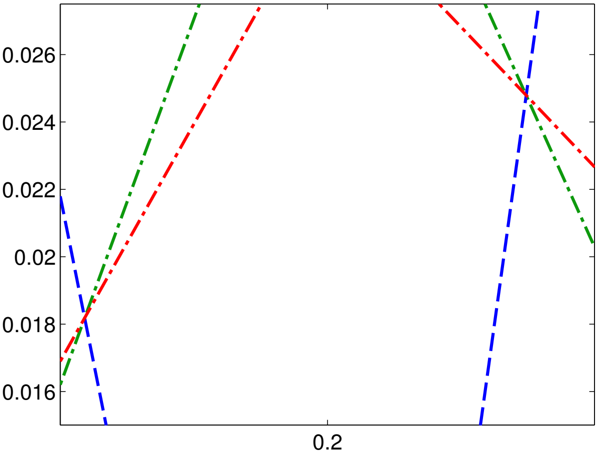
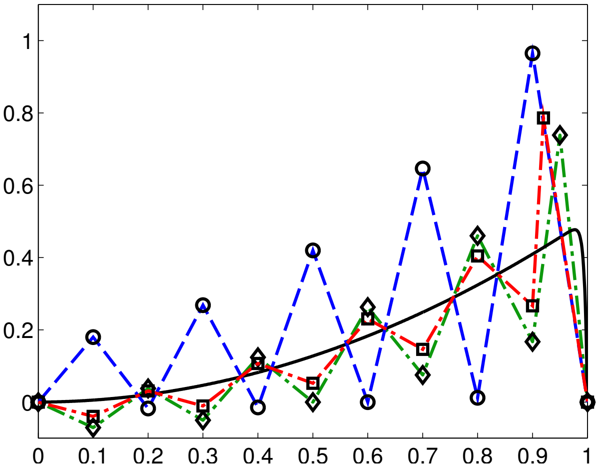


Blow-up of part of previous graph



All computed solutions intersect at points that are independent of where one adds mesh point(s) in $(0.9, 1)$

True solution and 3 computed solutions



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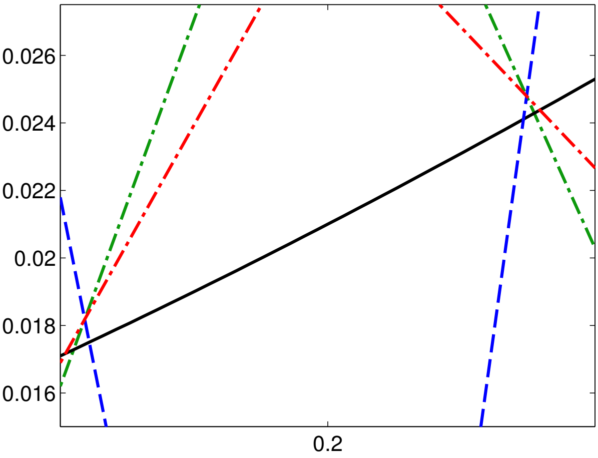

Same behaviour occurs for all N and ε (provided $N \ll \varepsilon^{-1}$) and when other convection-diffusion test problems are considered.

Theoretical explanation

SYZ07 give a complete theoretical explanation of the two phenomena that we observed:

1. Common intersection points of all piecewise linear Galerkin solutions when extra mesh point(s) are added inside the mesh interval containing the layer
2. These common intersection points lie close to the true solution

But their analysis is only for the special case where $a(\cdot)$ is constant and $b(\cdot) \equiv 0$
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$\{u_i^{n+1}, i = 1, \ldots, n; u_{s_1}^{n+1}\}$ given by

(3.1)
$$\sum_{i=1}^{n-1} u_i^{n+1} a(\phi_i, \phi_j) + u_n^{n+1} a(\tilde{\phi}_n, \phi_j) + u_{s_1}^{n+1} a(\phi_{s_1}, \phi_j) = (f, \phi_j), \quad j = 1, 2, \ldots, n-1,$$

$$\sum_{i=1}^{n-1} u_i^{n+1} a(\phi_i, \tilde{\phi}_n) + u_n^{n+1} a(\tilde{\phi}_n, \tilde{\phi}_n) + u_{s_1}^{n+1} a(\phi_{s_1}, \tilde{\phi}_n) = (f, \tilde{\phi}_n), \tag{3.2}$$

and

$$\sum_{i=1}^{n-1} u_i^{n+1} a(\phi_i, \phi_{s_1}) + u_n^{n+1} a(\tilde{\phi}_n, \phi_{s_1}) + u_{s_1}^{n+1} a(\phi_{s_1}, \phi_{s_1}) = (f, \phi_{s_1}). \tag{3.3}$$

Note that for $1 \le j \le n-1$, $a(\tilde{\phi}_n, \phi_j) = a(\phi_n, \phi_j)$ and $a(\phi_{s_1}, \phi_j) = 0$, and (3.1) leads to

$$\sum_{i=1}^{n} u_i^{n+1} a(\phi_i, \phi_j) = (f, \phi_j), \quad j = 1, 2, \ldots, n-1. \tag{3.4}$$

On the other hand, for $1 \le i \le n-1$, $a(\phi_i, \tilde{\phi}_n) = a(\phi_i, \phi_n)$, and (3.2) yields

$$\sum_{i=1}^{n-1} u_i^{n+1} a(\phi_i, \phi_n) + u_n^{n+1} a(\tilde{\phi}_n, \tilde{\phi}_n) = (f, \tilde{\phi}_n) - u_{s_1}^{n+1} a(\phi_{s_1}, \tilde{\phi}_n). \tag{3.5}$$

For $1 \le i \le n-1$, $a(\phi_i, \phi_{s_1}) = 0$, so it follows from (3.3),

$$u_n^{n+1} a(\tilde{\phi}_n, \phi_{s_1}) = (f, \phi_{s_1}) - u_{s_1}^{n+1} a(\phi_{s_1}, \phi_{s_1}). \tag{3.6}$$

Let $p = (1-s)/h_{n+1}$. Observe $\phi_n = \tilde{\phi}_n + p\phi_{s_1}$. Combining two equations above according to (3.5)+$p*$(3.6), we have

$$\sum_{i=1}^{n-1} u_i^{n+1} a(\phi_i, \phi_n) + u_n^{n+1} a(\tilde{\phi}_n, \phi_n) = (f, \phi_n) - u_{s_1}^{n+1} a(\phi_{s_1}, \phi_n). \tag{3.7}$$

Hence,

Simpler proof of common intersection points (general a, b)

Solve problem using the piecewise linear Galerkin method on a uniform mesh with N subintervals, where $N \ll \varepsilon^{-1}$. Set $h = 1/N$. Denote the computed solution by $u_h \in C[0, 1]$.

Boundary layer lies in the interval $(1 - h, 1)$ because $N \ll \varepsilon^{-1}$.

Introduce an arbitrary additional mesh point (or points) in the interval $(1 - h, 1)$. Let $\hat{u}_h$ denote the piecewise linear Galerkin solution computed on this modified mesh.

We shall show that $u_h = \hat{u}_h$ at one *intersection point* in each interval $(h, 2h)$, $(2h, 3h), \ldots, (1 - 2h, 1 - h)$; furthermore, these intersection points are independent of the additional mesh point(s) in $(1 - h, 1)$.

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On the interval $[0, 1-h]$, the computed solution u_h is the piecewise linear Galerkin solution of the two-point boundary problem

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and the computed solution $\hat{u}_h$ is the piecewise linear Galerkin solution of the boundary value problem

$$-\varepsilon w'' + aw' + bw = f \text{ on } (0, 1-h); \quad w(0) = 0, \; w(1-h) = \hat{u}_h(1-h).$$

Consequently their difference $u_h - \hat{u}_h$ is the piecewise linear Galerkin solution of the boundary value problem

$$-\varepsilon z'' + az' + bz = 0 \quad \text{on } (0, 1-h),$$
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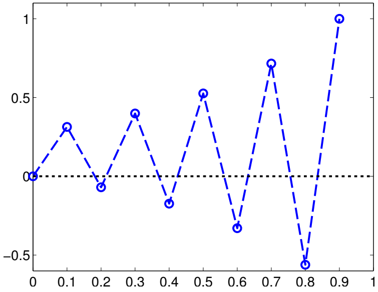
Lemma (to be proved later)

On the mesh $\{0, h, 2h, \ldots, 1-h\}$, p.w.linear Galerkin solution of

$$-\varepsilon\zeta'' + a\zeta' + b\zeta = 0 \text{ on } (0, 1-h), \quad \zeta(0) = 0, \quad \zeta(1-h) = 1$$

has a zero in each interval $(h, 2h), (2h, 3h), \ldots, (1-2h, 1-h)$ and is otherwise non-zero in $(0, 1-h]$. Example with $h = 1/10$:



These zero points will be the intersection points that we seek.

$u_h = \hat{u}_h$ at a set of points that is independent of $[1-h, 1]$

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also equals zero at these points,
i.e., $u_h = \hat{u}_h$ at these *intersection points*.

Note also that the intersection points depend only on the problem

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Lemma

Consider the two-point boundary value problem

$$-\varepsilon\zeta'' + a\zeta' + b\zeta = 0 \text{ on } (0, 1-h), \quad \zeta(0) = 0, \quad \zeta(1-h) = 1.$$

Subdivide the interval $[0, 1-h]$ *by a uniform mesh with intervals of width* h *and assume that*

$$\min_{[0,1]} \left(\frac{a}{2} - \left| \frac{hb}{6} - \frac{\varepsilon}{h} \right| \right) > 0. \tag{1}$$

Then the p.w. linear Galerkin solution of this problem oscillates about zero, i.e., the computed solution equals zero at one point in each of the mesh intervals $(h, 2h)$, $(2h, 3h), \ldots, (1-2h, 1-h)$ *and is otherwise non-zero in* $(0, 1-h]$.

Proof of Lemma

Let $g \in C[0, 1-h]$ denote the piecewise linear Galerkin solution of the given problem on the given mesh. The difference scheme defining the nodal values of g is (because of our quadrature rule)

$$-\frac{\varepsilon}{h^2}(g_{i+1} - 2g_i + g_{i-1}) + \frac{a_i(g_{i+1} - g_{i-1})}{2h} + \frac{b_i}{6}(g_{i+1} + 4g_i + g_{i-1}) = 0$$

for $i = 1, \ldots, N-2$, with $g_0 = 0$ and $g_{N-1} = 1$, where $g_j := g(jh)$ for all j. This scheme can be rewritten as

$$\left(\frac{a_i}{2h} + \frac{b_i}{6} - \frac{\varepsilon}{h^2}\right) g_{i+1} + \left(\frac{4b_i}{6} + \frac{2\varepsilon}{h^2}\right) g_i + \left(-\frac{a_i}{2h} - \frac{\varepsilon}{h^2} + \frac{b_i}{6}\right) g_{i-1} = 0$$

for $i = 1, \ldots, N-2$.
The hypothesis (1) ensures that the coefficients of g_{i+1} and g_i are positive but the coefficient of g_{i-1} is negative.

Proof of Lemma (continued)

Write the scheme as

$$\alpha_i g_{i+1} + \beta_i g_i - \gamma_i g_{i-1} = 0, \quad \text{with } g_0 = 0, \; g_{N-1} = 1 \qquad (2)$$

and $\alpha_i > 0, \beta_i > 0, \gamma_i > 0$.
The solution of this difference scheme cannot have $g_1 = 0$ because then taking $i = 1$ would imply that $g_2 = 0$, and a similar inductive argument then leads to $g_{N-1} = 0$ which is false. Thus $g_1 \neq 0$.

If $g_1 > 0$, then taking $i = 1$ in (2) and recalling the signs of the coefficients there and $g_0 = 0$, we see that $g_2 < 0$. Similarly, $g_1 < 0$ implies that $g_2 > 0$. Thus in all cases one has $g_1 g_2 < 0$. One can now proceed inductively, invoking (2) for $i = 2, 3, \ldots, N-2$ and using the signs of its coefficients, to get $g_i g_{i+1} < 0$ for each i. The desired result follows.

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Accuracy of computed solution at the intersection points

Theorem

Subdivide $[0,1]$ *by a uniform mesh of width* h*. Assume that* $h \geq \varepsilon |\ln \varepsilon|$ *and that* (1) *is satisfied. Then the piecewise linear Galerkin solution* u_h *of the original two-point boundary value problem on the mesh* $\{0, h, 2h, \ldots, 1-h, 1\}$ *satisfies*

$$|u(\zeta_i) - u_h(\zeta_i)| \leq Ch^2 \quad \textit{for } i = 2, 3, \ldots, N-1,$$

where the ζ_i *are the intersection points and the constant* C *is independent of* ε *and* h*.*

Proof of Theorem

Since $h \geq \varepsilon \, |\ln \varepsilon|$, one can insert extra mesh points in the interval $(1-h, 1)$ to construct a Bakhvalov mesh for the original problem. It follows from work of Andreev and Kopteva that the piecewise linear Galerkin solution u_B on the Bakhvalov mesh satisfies $\max_{[0,1]} |u(x) - u_B(x)| \leq Ch^2$ for some constant C.
In particular this implies that $|u(\zeta_i) - u_B(\zeta_i)| \leq Ch^2$ for each i (since (1) holds true by hypothesis, our Lemma is valid and consequently the ζ_i are well defined). But our earlier analysis showed that $u_h(\zeta_i) = u_B(\zeta_i)$ for each i, so we are done.

Remark

The idea of this proof comes from SYZ07. But the Andreev and Kopteva result is valid only for certain difference schemes including our scheme — this motivates the unusual quadrature rule that we introduced into the Galerkin method (this quadrature rule does not appear in SYZ07 because only the case $a(\cdot)$ *constant and* $b \equiv 0$ *is considered there).*

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In particular this implies that $|u(\zeta_i) - u_B(\zeta_i)| \leq Ch^2$ for each i (since (1) holds true by hypothesis, our Lemma is valid and consequently the ζ_i are well defined). But our earlier analysis showed that $u_h(\zeta_i) = u_B(\zeta_i)$ for each i, so we are done.

Remark

The idea of this proof comes from SYZ07. But the Andreev and Kopteva result is valid only for certain difference schemes including our scheme — this motivates the unusual quadrature rule that we introduced into the Galerkin method (this quadrature rule does not appear in SYZ07 because only the case $a(\cdot)$ constant and $b \equiv 0$ is considered there).

Numerical method

Corollary

The piecewise linear interpolant $\widetilde{u}_h$ *of* $u_h(\zeta_i),\ i = 2, \ldots, N-1$, *satisfies*

$$\|u - \widetilde{u}_h\|_{L_\infty[0,1-\zeta_{N-1}]} \leq Ch^2.$$

See conference proceedings paper for a numerical example.

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