An adaptive *hp*-discontinuous Galerkin approach for nonlinear convection-diffusion problems

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Introduction

Discontinuous Galerkin method (DGM)

- DG discretization of nonlinear convection-diffusion problems
- leads to a nonlinear algebraic system,
  - solve by an iterative method,
- *hp*-adaptive method
  - *residuum nonconformity estimator* : element for refinement?
    = measure of residuum + measure of nonconformity
  - *regularity indicator* : *h*- or *p*- adaptation?
- stopping criterion for the iterative method

Problem definition

Scalar nonlinear convection-diffusion equation

$$\begin{aligned} \nabla \cdot \vec{f}(u) - \nabla \cdot (\mathbf{K}(u)\nabla u) &= g && \text{in } \Omega, \\ u &= u_D && \text{on } \partial\Omega_D, \\ \nabla u \cdot \mathbf{n} &= g_N && \text{on } \partial\Omega_N, \end{aligned} \tag{1}$$

- $u : \Omega \to \mathbb{R}$, $\Omega \subset \mathbb{R}^d$,
- $f : \mathbb{R} \to \mathbb{R}^d$ – nonlinear convection,
- $\mathbf{K} : \mathbb{R} \to \mathbb{R}^d \times \mathbb{R}^d$ – nonlinear diffusion,
- $g : \Omega \to \mathbb{R}$ – source terms,
- $u_D : \partial\Omega_D \to \mathbb{R}$ – Dirichlet boundary condition,
- $g_N : \partial\Omega_N \to \mathbb{R}$ – Neumann boundary condition,

  we assume that there exists a unique weak solution.

Discretization

Triangulation

- let $\mathscr{T}_h$, $h > 0$ be a partition of $\overline{\Omega}$
- $\mathscr{T}_h = \{K\}_{K \in \mathscr{T}_h}$, K are d-dimensional polyhedra,

Functional spaces

- *broken Sobolev space*

$$H^s(\Omega, \mathscr{T}_h) := \{v; v|_K \in H^s(K)\ \forall\, K \in \mathscr{T}_h\}, \quad s \geq 1,$$

- let $\mathrm{p} = \{p_K,\ K \in \mathscr{T}_h\}, \quad p_K \geq 0,$ integer $\forall K \in \mathscr{T}_h$
- space of *piecewise polynomial functions*

$$S_{hp} := \{v;\ v \in L^2(\Omega),\ v|_K \in P_{p_K}(K)\ \forall K \in \mathscr{T}_h\},$$

DG discretization

- interior penalty Galerkin (IPG) variants of DG
- form $\tilde{c}_h(\cdot,\cdot) : H^2(\Omega, \mathscr{T}_h) \times H^2(\Omega, \mathscr{T}_h) \to \mathbb{R}$

Discrete problem

We seek $u_h \in S_{hp}$ such that

$$\tilde{c}_h(u_h, \varphi_h) = 0 \qquad \forall \varphi_h \in S_{hp} \tag{2}$$

- nonlinear algebraic system
- solve inexactly by an iterative Newton-like method

Equivalence of the (exact) weak solution

Nonconformity $\mathcal{N}_h : H^1(\Omega, \mathscr{T}_h) \to \mathbb{R}$

$$\mathcal{N}_h(v)^2 := 2 \sum_{\Gamma \in \mathscr{F}_h^I} \int_\Gamma h_\Gamma^{-1} [\![v]\!]^2 \,\mathrm{d}S + \sum_{\Gamma \in \mathscr{F}_h^D} \int_\Gamma h_\Gamma^{-1} (v - u_D)^2 \,\mathrm{d}S, \quad (3)$$

Lemma (chracterization of the exact solution)

i) Let $u \in H^2(\Omega)$ be the weak solution of (1) then

$$\tilde{c}_h(u, v_h) = 0 \quad \forall v_h \in H^2(\Omega, \mathscr{T}_h) \quad \& \quad \mathcal{N}_h(u) = 0. \quad (4)$$

ii) If $u \in H^2(\Omega, \mathscr{T}_h)$ satisfies both conditions of (4) then u is the weak solution of (1).

estimates of $u - u_h$: violation of (4)

Functional representation: $\tilde{c}_h(u_h, \varphi_h) = 0 \ \forall \varphi_h \in S_{hp}$

Functional spaces

- let $X := H^2(\Omega, \mathscr{T}_h)$, $\quad u, u_h \in X$, $\quad \|\cdot\|_X$,
- let $A_h : X \to X'$: $\quad \langle A_h w, \varphi \rangle := \tilde{c}_h(w, \varphi) \quad \forall w, \varphi \in X$
- $\|A_h w\|_{X'} := \sup_{\varphi \in X, \varphi \neq 0} \frac{\langle A_h w, \varphi \rangle}{\|\varphi\|_X}$.

Residuum definition

- if u is the exact solution then $A_h u = 0$
- let $u_h \in S_{hp}$ be approximate solution

$$\mathcal{R}_h(u_h) := \|A_h u_h - A_h u\|_{X'} = \sup_{v \in X} \frac{\langle A_h u_h, v \rangle}{\|v\|_X} = \sup_{v \in X} \frac{\tilde{c}_h(u_h, v)}{\|v\|_X}$$

residuum error in dual norm

Error measure of $u - u_h$

First building block

residuum error in the dual norm $\mathcal{R}_h(u_h) := \sup_{v \in X} \frac{\tilde{c}_h(u_h, v)}{\|v\|_X}$

Second building block

nonconformity

$\mathcal{N}_h(u_h)^2 := 2 \sum_{\Gamma \in \mathscr{F}_h^I} \int_\Gamma h_\Gamma^{-1} [\![u_h]\!]^2 \, \mathrm{d}S + \sum_{\Gamma \in \mathscr{F}_h^D} \int_\Gamma h_\Gamma^{-1} (u_h - u_D)^2 \, \mathrm{d}S$

Error measure

$\mathcal{E}_h(u_h) := (\mathcal{R}_h(u_h)^2 + \mathcal{N}_h(u_h)^2)^{1/2} > 0$

Lemma

$\mathcal{E}_h(u_h) = 0 \iff u_h = u$

Estimate

? $\mathcal{R}_h(u_h) \leq \rho_h(u_h)$?

Residuum error estimation: $\mathcal{R}_h(u_h) := \sup_{v \in X, v \neq 0} \tilde{c}_h(u_h, v)/\|v\|_X$

Finite dimensional subspaces of X

- let $K \in \mathscr{T}_h$, $p = 0, 1, 2, \ldots$
  $S_K^p := \{\phi_h : \ \phi_h|_K \in P^p(K), \ \phi_h = 0 \ \text{ on } \Omega \setminus K\},$
- $S_{hp} = \mathrm{span}\{S_K^{p_K}, \ K \in \mathscr{T}_h\}$
- $S_{hp}^+ := \mathrm{span}\{S_K^{p_K+1}, \ K \in \mathscr{T}_h\}$

global residuum estimator: $$\rho_h(u_h) := \sup_{v_h \in S_{hp}^+} \frac{\tilde{c}_h(u_h, v_h)}{\|v_h\|_X}$$

element residuum estimator: $$\rho_{h,K}(u_h) := \sup_{v_h \in S_K^{p_K+1}} \frac{\tilde{c}_h(u_h, v_h)}{\|v_h\|_X}$$

algebraic residuum estimator: $$\rho_h^A(u_h) := \sup_{v_h \in S_{hp}} \frac{\tilde{c}_h(u_h, v_h)}{\|v_h\|_X}$$

Evaluation of residuum estimators

Lemma

Let $((\cdot,\cdot))_X$ be a scalar product generating the $\|\cdot\|_X$-norm and let

$$((\phi_K,\phi_{K'}))_X = 0 \quad \forall \phi_K \in S_K^p \ \forall \phi_{K'} \in S_{K'}^p,\ K \neq K',\ p \geq 0$$

then
$$\rho_h(u_h)^2 = \sum_{K\in\mathscr{T}_h} \rho_{h,K}(u_h)^2.$$

Definition of $\|\cdot\|_X$

$$\|v\|_X := \left(d\|v\|^2_{L^2(\Omega)} + \varepsilon|v|^2_{H^1(\Omega)}\right)^{1/2}, \qquad \varepsilon > 0, d > 0$$

Evaluation $\rho_{h,K}$

task of seeking of a constrain extrema in $S_K^{p_K+1}$

Stopping criterion for the Newton-like method:

Estimators

global residuum estimator: $\rho_h(u_h) := \sup_{v_h \in S_{hp}^+} \frac{\tilde{c}_h(u_h, v_h)}{\|v_h\|_X}$

algebraic residuum estimator: $\rho_h^A(u_h) := \sup_{v_h \in S_{hp}} \frac{\tilde{c}_h(u_h, v_h)}{\|v_h\|_X}$

Algebraic residuum

- u_h is approximate solution $\rho_h^A(u_h) = 0$,
- $\tilde{u}_h$ is output of the Newton-like method:

$$0 \leq \rho_h^A(\tilde{u}_h) \leq \rho_h(\tilde{u}_h) \leq \mathrm{tol}$$

- *stopping criterion*: $\rho_h^A(\tilde{u}_h) \leq \beta \rho_h(\tilde{u}_h)$, $\beta < 1$ $\beta \approx 10^{-2}$

Regularity indicator

Proposal of the regularity estimator

$$g_K(u_h) := \frac{\int_{\partial K \cap \Omega} [\![u_h]\!]^2 \, \mathrm{d}S}{|K| \, h_K^{2p_K - 2}}, \quad K \in \mathscr{T}_h \tag{5}$$

u "is regular" $\Rightarrow g_K(u_h) = O(h_K)$,

u "is not regular" $\Rightarrow g_K(u_h) = O(h_K^{2\delta-1})$, $\delta < 0$

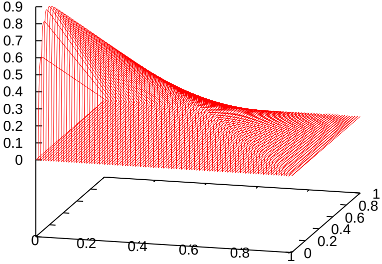
Regularity criterion

$g_K(u_h) \leq 1 \quad \Rightarrow$ solution is regular $\Rightarrow$ p-refinement,

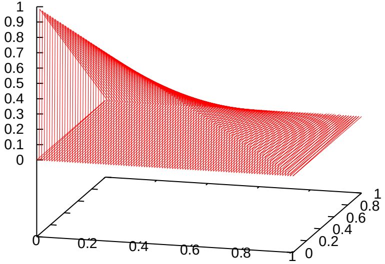
$g_K(u_h) > 1 \quad \Rightarrow$ solution is not regular $\Rightarrow$ h-refinement,

Scalar linear equation

- scalar linear convection-diffusion equation
  $$-\varepsilon\Delta u - \frac{\partial u}{\partial x_1} - \frac{\partial u}{\partial x_2} = g \quad \text{in}\,(0,1)^2,\ \varepsilon = 10^{-2},\ \varepsilon = 10^{-3}$$
- exact: $u = \left(c_1 + c_2(1-x_1) + e^{-x_1/\varepsilon}\right)\left(c_1 + c_2(1-x_2) + e^{-x_2/\varepsilon}\right)$,
  $c_1 = -\exp(-1/\varepsilon)$, $c_2 = -1 - c_1$



$\varepsilon = 10^{-2}$



$\varepsilon = 10^{-3}$

initial mesh $h = 1/8$, P_1 approximation

BL: Error and estimators convergence

$\varepsilon = 10^{-2}$

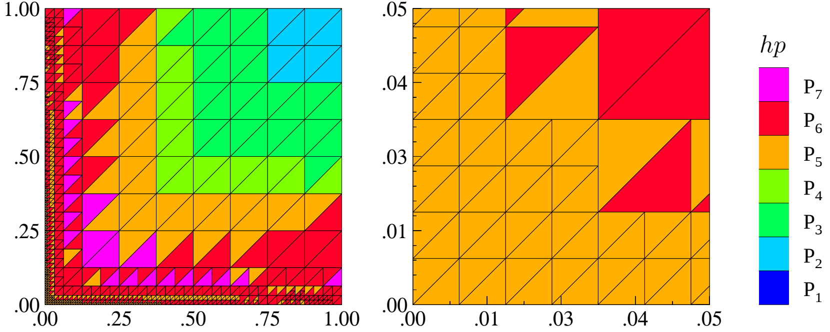
lev	$\#\mathscr{T}_h$	N_h	$\|u-\tilde{u}_h\|_X^2$	EOC	$\mathcal{N}_h(\tilde{u}_h)^2$	EOC	$\rho_h(\tilde{u}_h)^2$	EOC	i_{eff}	CPU(s)
0	128	384	3.93E-01	–	1.12E+00	–	1.04E+00	–	1.29	0.3
1	128	768	3.91E-01	0.01	8.45E-01	0.82	6.09E-01	1.55	1.12	0.4
2	128	1248	2.52E-01	1.82	5.67E-01	1.65	3.41E-01	2.40	1.07	0.5
3	158	1968	1.21E-01	3.23	2.82E-01	3.06	1.63E-01	3.25	1.06	0.9
4	236	3450	3.72E-02	4.18	8.55E-02	4.26	4.83E-02	4.32	1.05	1.4
5	380	6322	6.93E-03	5.55	1.50E-02	5.75	7.41E-03	6.19	1.01	2.8
6	554	10492	7.86E-04	8.60	1.66E-03	8.67	8.40E-04	8.60	1.01	5.3
7	776	17270	5.73E-05	10.51	1.14E-04	10.76	5.62E-05	10.85	1.00	9.7
8	938	22438	2.07E-05	7.79	4.28E-05	7.47	2.20E-05	7.15	1.01	15.8

$\varepsilon = 10^{-3}$

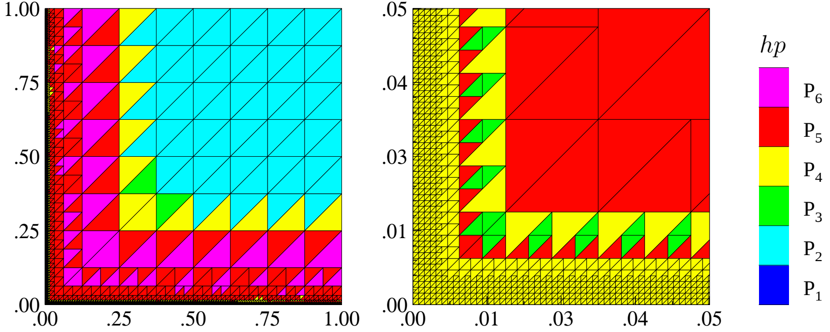
lev	$\#\mathscr{T}_h$	N_h	$\|u-\tilde{u}_h\|_X^2$	EOC	$\mathcal{N}_h(\tilde{u}_h)^2$	EOC	$\rho_h(\tilde{u}_h)^2$	EOC	i_{eff}	CPU(s)
0	128	384	2.62E-02	–	2.04E+00	–	6.47E-01	–	1.05	0.3
1	128	768	5.15E-01	-8.59	2.02E+00	0.02	5.28E-01	0.59	1.00	0.4
2	146	1076	5.27E-01	-0.13	2.46E+00	-1.15	6.20E-01	-0.95	1.01	0.5
3	206	1834	4.53E-01	0.57	2.83E+00	-0.53	6.62E-01	-0.25	1.01	0.8
4	368	4076	3.89E-01	0.38	2.72E+00	0.10	5.46E-01	0.48	1.01	1.5
5	920	10772	3.04E-01	0.51	2.20E+00	0.43	4.20E-01	0.54	1.01	3.0
6	1916	22610	1.54E-01	1.83	1.12E+00	1.83	2.06E-01	1.91	1.01	7.4
7	3794	44138	4.80E-02	3.49	3.33E-01	3.62	6.06E-02	3.67	1.01	18.7
8	6416	75180	9.32E-03	6.15	6.20E-02	6.31	1.14E-02	6.27	1.01	36.0
9	8078	108732	1.36E-03	10.44	9.46E-03	10.19	1.75E-03	10.16	1.01	65.7

$$i_{\mathrm{eff}} = \frac{\rho_h(\tilde{u}_h)^2 + \mathcal{N}_h(\tilde{u}_h)^2}{\|u-\tilde{u}_h\|_X^2 + \mathcal{N}_h(\tilde{u}_h)^2}$$

BL, $\varepsilon = 10^{-2}$, final *hp*-mesh

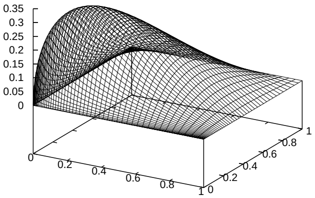


BL, $\varepsilon = 10^{-3}$, final *hp*-mesh



Scalar nonlinear equation

- scalar nonlinear convection-diffusion equation
  $$-\nabla \cdot (\mathbf{K}(u)\nabla u) - \frac{\partial u^2}{\partial x_1} - \frac{\partial u^2}{\partial x_2} = g \quad \text{in}\,(0,1)^2$$
- $$\mathbf{K}(u) = \varepsilon \begin{pmatrix} 2+\arctan(u) & (2-\arctan(u))/4 \\ 0 & (4+\arctan(u))/2 \end{pmatrix}, \qquad \varepsilon = 10^{-3}$$
- exact: $u = (x_1^2 + x_2^2)^{-3/4} x_1 x_2 (1-x_1)(1-x_2) \in H^{3/2-\epsilon}(\Omega)$



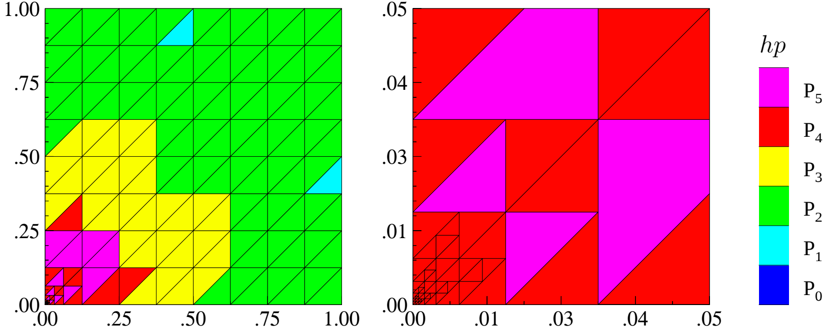
- initial mesh $h = 1/8$
- P_1 approximation

Singul: Error convergence

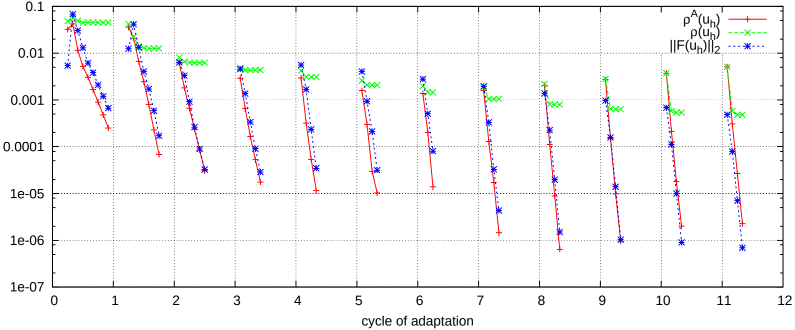
lev	$\#\mathscr{T}_h$	N_h	$\|u-\tilde{u}_h\|_X^2$	EOC	$\mathcal{N}_h(\tilde{u}_h)^2$	EOC	$\rho_h(\tilde{u}_h)^2$	EOC	$i_{\rm eff}$	CPU(s)
0	128	384	1.32E-02	–	1.41E-01	–	4.52E-02	–	1.05	0.5
1	128	759	5.98E-03	2.32	6.70E-02	2.18	1.26E-02	3.75	1.01	0.8
2	128	919	5.50E-03	0.87	6.36E-02	0.55	6.26E-03	7.31	1.00	1.1
3	128	969	4.30E-03	9.31	5.52E-02	5.36	4.34E-03	13.81	1.00	1.4
4	134	1089	2.98E-03	6.29	3.96E-02	5.69	3.09E-03	5.86	1.00	1.6
5	140	1191	2.10E-03	7.81	2.75E-02	8.14	2.07E-03	8.91	1.00	1.9
6	152	1371	1.49E-03	4.82	1.93E-02	4.99	1.45E-03	5.03	1.00	2.1
7	158	1476	1.07E-03	9.07	1.37E-02	9.33	1.05E-03	8.72	1.00	2.5
8	164	1578	7.71E-04	9.78	9.78E-03	10.13	7.97E-04	8.34	1.00	2.9
9	176	1758	5.65E-04	5.75	7.04E-03	6.09	6.35E-04	4.21	1.00	3.3
10	188	1938	4.26E-04	5.81	5.15E-03	6.41	5.37E-04	3.43	1.00	3.8
11	200	2118	3.35E-04	5.41	3.87E-03	6.41	4.81E-04	2.48	1.00	4.3

$$i_{\rm eff} = \frac{\rho_h(\tilde{u}_h)^2+\mathcal{N}_h(\tilde{u}_h)^2}{\|u-\tilde{u}_h\|_X^2+\mathcal{N}_h(\tilde{u}_h)^2}$$

Singul, final *hp*-mesh



Singul, convergence of the iterative solver



algebraic indicator, global residuum indicator, (standard) algebraic residuum

stopping condition for the iterative solver: $\rho_h^A(\tilde{u}_h) \leq 10^{-2}\rho_h(\tilde{u}_h)$

Conclusion

Summary

- *hp*-DGFEM for nonlinear convection-diffusion problems
- residuum nonconformity indicator and regularity estimator
- no theoretical justification
- reasonable computational properties

Outlook

- theoretical justification
- application to more challenging problems