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A STRENGTHENING OF THE POINCARÉ RECURRENCE THEOREM ON MV-ALGEBRAS

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Abstract

The strong version of the Poincaré recurrence theorem states that for any probability space $(\Omega, \mathcal{S}, P)$, any P-measure preserving transformation $T : \Omega \to \Omega$ and any $A \in \mathcal{S}$ almost every point of A returns to A infinitely many times. In [8] (see also [4]) the theorem has been proved for MV-algebras of some type. The present paper contains a remarkable strengthening of the result stated in [8].

1. Introduction

The Poincaré recurrence theorem [5] has been proved for Boolean algebras [7], topological spaces [2] and for MV-algebras of some types in [8]. Recall that MV-algebras play an analogous role in multivalued logics as Boolean algebras in two valued logics. Any MV-algebra can be simply characterized by the help of an l-group as an interval in it.

An l-group is an algebraic structure $(G, +, \leq)$, where $(G, +)$ is a commutative group, $(G, \leq)$ is a lattice, and the implication $a \leq b \implies a + c \leq b + c$ holds. MV-algebra is an algebraic structure

$$(M, 0, u, \neg, \oplus, \odot),$$

where 0 is the neutral element in G, u is a positive element, $M = \{x \in G; 0 \leq x \leq u\}$, $\neg : M \to M$ is a unary operation given by the equality

$$\neg x = u - x,$$

and $\oplus, \odot$ are two binary operations given by

$$a \oplus b = (a + b) \wedge u,$$

$$a \odot b = (a + b - u) \vee 0.$$

Example 1. Let $\mathcal{S}$ be an algebra of subsets of a set Ω. Then $\mathcal{S}$ is an MV-algebra. If we identify sets A with their characteristic functions, then the corresponding l-group $(G, +, \leq)$ consists of all measurable functions, $+$ is the sum of functions, $\leq$ corresponds to the set inclusion. Then $0 = 0_\Omega, u = 1_\Omega$,

$$\neg\chi_A = \chi_{A^c} = 1_\Omega - \chi_A,$$

$$\chi_A \oplus \chi_B = \chi_{A\cup B},$$

$$\chi_A \odot \chi_B = \chi_{A\cap B}.$$

Example 2. Let $[0,1]$ be the unit interval in the set R of real numbers. Then $(R, +, \leq)$ is an l-group, so that $[0,1]$ is an MV-algebra

$$\neg a = 1 - a,$$

$$a \oplus b = \min(a + b, 1),$$

$$a \odot b = \max(a + b - 1, 0).$$

In the following definitions we shall use the symbols $a_n \nearrow a$ and $b_n \searrow b$. It means that $a_n \leq a_{n+1}, b_n \geq b_{n+1}, n = 1, 2, \ldots$ and $a = \bigvee_{n=1}^\infty a_n, b = \bigwedge_{n=1}^\infty b_n$.

Definition 1. A σ-complete MV-algebra is called weakly σ-distributive, if for any double sequence $(a_{ij})_{ij}$ of elements of M such that

$$a_{ij} \searrow 0 (j \to \infty)$$

there holds

$$\bigwedge_{\phi: N\to N} \bigvee_{j=1}^{\infty} a_{i\phi(i)} = 0.$$

(The name distributive is motivated by the equality

$$\bigwedge_{\phi: N\to N} \bigvee_{j=1}^{\infty} a_{i\phi(i)} = \bigvee_{i=1}^{\infty} \bigwedge_{j=1}^{\infty} a_{ij} = 0.)$$

Definition 2. An MV-algebra with product is an MV-algebra with a commutative and associative binary operation $\star$ satisfying the following conditions (see [6], equivalently [3]):

(i) $a \star u = a$,

(ii) $a \star (b \oplus c) = (a \star b) \oplus (a \star c)$

(iii) $a_n \nearrow a \Longrightarrow a_n \star b \nearrow a \star b$.

Definition 3. A state on an MV-algebra M is a mapping $m : M \to [0,1]$ satisfying the following conditions:

(i) $m(u) = 1, m(0) = 0$,

(ii) $a \odot b = 0 \Longrightarrow m(a \oplus b) = m(a) + m(b)$,

(iii) $a_n \nearrow a \Longrightarrow m(a_n) \nearrow m(a)$.

Definition 4. Let M be a σ-complete MV-algebra with product, $m : M \to [0,1]$ be a state. By an m-preserving transformation of M we understand a mapping $\tau : M \to M$ satisfying the following conditions:

(i) $\tau(u) = u, \tau(0) = 0$;

(ii) $\tau(a \odot b) = \tau(a) \odot \tau(b)$;

(iii) $\tau(a \oplus b) = \tau(a) \oplus \tau(b)$;

(iv) $\tau(a \star b) = \tau(a) \star \tau(b)$;

(v) $\tau(a \vee b) = \tau(a) \vee \tau(b)$;

(vi) $\tau(a \wedge b) = \tau(a) \wedge \tau(b)$;

(vii) $a_n \nearrow a \Longrightarrow \tau(a_n) \nearrow \tau(a)$;

(viii) $m(\tau(a)) = m(a)$.

The following theorem has been proved in [8]. In the following text we use the notation text $\Pi_{i=k}^{\infty} c_i = \bigwedge_{j=1}^{\infty} (c_k \star c_{k+1} \star ... \star c_{k+j})$.

Theorem 1. Let $(M, \star)$ be a σ-complete weakly σ-distributive MV-algebra with product, $m : M \to [0,1]$ be a state, $\tau : M \to M$ be a measure preserving transformation. Then

$$m\left(\bigvee_{k=1}^{\infty} a \star \Pi_{i=k}^{\infty} \tau^i(\neg a)\right) = \lim_{k\to\infty} m\left(a \star \Pi_{i=k}^{\infty} \tau^i(\neg a)\right) = 0.$$

2. Strong Poincaré recurrence theorem

The following theorem is a strengthening of Theorem 1. The proof of the theorem is new, too.

Theorem 2. Let $(M, \star)$ be a σ-complete MV-algebra with product. Let $m : M \to [0,1]$ satisfy the following conditions:

1. $m(0) = 0$,
2. $a \leq b \Longrightarrow m(a) \leq m(b)$,
3. $a \odot b = 0 \Longrightarrow m(a \oplus b) = m(a) + m(b)$.

Let $\tau : M \to M$ satisfy the conditions

4. $\tau(0) = 0$,
5. $a \leq b \Longrightarrow \tau(a) \leq \tau(b)$,
6. $\tau(a \odot b) = \tau(a) \odot \tau(b)$,
7. $m(\tau(a)) = m(a)$ for all $a \in M$.

Then there holds for any $a \in M$ and any $k \in N$

$$m\left(a \star \Pi_{i=k}^{\infty} \tau(\neg a)\right) = 0.$$

(Here $\Pi_{i=k}^{\infty} c_i = \bigwedge_{j=0}^{\infty} \Pi_{i=k}^{k+j} c_i, \Pi_{i=k}^{k+j} c_i = c_k \star c_{k+1} \star \ldots \star c_{k+j}$.)

Proof. Let $a \in M$. Put

$$b = a \star \tau(\neg a) \star \tau^2(\neg a) \star \cdots \star \tau^n(\neg a) \star \cdots = a \star \bigwedge_{n=1}^{\infty} \Pi_{i=1}^{n} \tau^i(\neg a).$$

We have

$$\begin{aligned} &b \leq a, \\ &b \leq \tau^n(\neg a). \end{aligned}$$

Then

$$\tau^n(b) \leq \tau^n(a), b \leq \tau^n(\neg a),$$

hence

$$b \odot \tau^n(b) \leq \tau^n(a) \odot \tau^n(\neg a) = \tau^n(a \odot \neg a) = \tau^n(0) = 0.$$

Also if $l, j \in N, l < j$, then

$$\tau^l(b) \odot \tau^j(b) = \tau^l(b \odot \tau^{(j-l)}(b)) = \tau^l(0) = 0.$$

We see that $(\tau^j(b))_{i=0}^{\infty}$ is a disjoint system, hence

$$\Sigma_{j=1}^{n} m(\tau^j(b)) = m(\oplus_{j=1}^{n} \tau^j(b)) \leq 1.$$

Of course, $m(\tau(b)) = m(b)$ for $j = 1, 2, \ldots, n$, hence

$$\Sigma_{j=1}^{n} m(\tau^j(b)) = \Sigma_{j=1}^{n} m(b) = nm(b).$$

From the relation

$$m(b) \leq \frac{1}{n}$$

for any $n \in N$ we obtain

$$0 = m(b) = m\left(a \star \Pi_{i=1}^{\infty} \tau(\neg a)\right). \qquad (\star)$$

If we use $s = \tau^k : M \to M$ instead of τ we obtain by $(\star)$

$$m(a \star \Pi_{i=k}^{\infty} \tau^i(\neg a)) \leq m(a \star \Pi_{i=1}^{\infty} (\tau^k)^i(\neg a))) = m(a \star \Pi_{i=1}^{\infty} s^i(\neg a)) = 0,$$

hence

$$\lim_{k\to\infty} m(a \star (\Pi_{j=k}^{\infty}(\tau^j(\neg a)))) = 0.$$

Corollary. Let m satisfy in addition the continuity condition

$$a_n \nearrow a \Longrightarrow m(a_n) \nearrow m(a).$$

Then

$$m(\bigvee_{k=1}^{\infty} a \star (\Pi_{j=k}^{\infty} \tau_j(\neg a))) = 0$$

3. Conservative mappings

P. R. Halmos [1] has shown that it is not necessary to assume that τ is measure preserving for the proof of the Poincaré theorem. It suffices to assume that there is no set A of positive measure such that the family $(\tau^i(A))_{i=1}^{\infty}$ is disjoint. We shall show that this works also in MV-algebras. Of course, instead of the family of sets of zero measure we shall consider an ideal $\mathcal{N} \subset \mathcal{M}$.

Definition 5. Let $\mathcal{M}$ be an MV-algebra with product. A subset $\mathcal{N} \subset \mathcal{M}$ is called a weak ideal if is satisfies the following conditions:

1. $0 \in \mathcal{N}$.
2. If $a \leq b, a \in \mathcal{M}, b \in \mathcal{N}$, then $a \in \mathcal{N}$.

A mapping $\tau : \mathcal{M} \to \mathcal{M}$ is called conservative if the following conditions hold:

3. If $(\tau^i(b))_{i=0}^{\infty}$ forms a disjoint system (i.e. $\tau^i(b) \odot \tau^j(b) = 0$ for $i \neq j$) then $b \in \mathcal{N}$.
4. $\tau(a \odot b) = \tau(a) \odot \tau(b)$ for any $a, b \in \mathcal{M}$.
5. $a \leq b$ implies $\tau(a) \leq \tau(b)$.
6. $b \in \mathcal{N} \Longleftrightarrow \tau(b) \in \mathcal{N}$.

Theorem 3. Let $\mathcal{M}$ be a σ-complete MV-algebra with product, $\mathcal{N} \subset \mathcal{M}$ be its weak ideal, $\tau : \mathcal{M} \to \mathcal{M}$ be a conservative mapping. Then

$$a \star \Pi_{i=k}^{\infty} \tau^i(\neg a) \in \mathcal{N}$$

for any $a \in \mathcal{M}$ and any $k \in N$.

Proof. Put

$$b = a \star \Pi_{i=1}^{\infty} \tau^i(\neg a).$$

Then

$$b \leq a$$
$$b \leq \tau^n(\neg a),$$

hence

$$b \odot \tau^n(b) \leq \tau^n(a \odot \neg a) = \tau^n(0) = 0.$$

It is easy to see that $(\tau^i(b))_{i=0}^{\infty}$ is a disjoint system, i.e.

$$\tau^i(b) \odot \tau^j(b) = 0$$

for $i \neq j$, hence

$$a \star (\Pi_{i=1}^{\infty} \tau^i(\neg a)) = b \in \mathcal{N}. \qquad (\star\star)$$

If τ is conservative, then also $s = \tau^k$ is conservative. Namely, if

$$s^i(b) \odot s^j(b) \in \mathcal{N}$$

for $i \neq j$ and $b \in \mathcal{M}$, then

$$\tau^i(c) \odot \tau^j(c) \in \mathcal{N}$$

for $i \neq j$ and $c = \tau^k(b)$. Therefore

$$\tau^k(b) = c \in \mathcal{N}$$

hence

$$b \in \mathcal{N}.$$

The equality $(\star\star)$ implies

$$a \star \Pi_{i=1}^{\infty} s^i(\neg a) \in \mathcal{N}$$

and since

$$a \star \Pi_{j=k}^{\infty} \tau^j(\neg a) \leq a \star \Pi_{i=i}^{\infty} \tau^{ki}(\neg a) = a \star \Pi_{i=1}^{\infty} s^i(\neg a) \in \mathcal{N}$$

we have

$$a \star \Pi_{j=k}^{\infty} \tau^j(\neg a) \in \mathcal{N}.$$

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