STEADY AND UNSTEADY 2D NUMERICAL SOLUTION OF GENERALIZED NEWTONIAN FLUIDS FLOW

Radka Keslerová, Karel Kozel

Departement of Technical Mathematics
Czech Technical University in Prague

Introduction

- Mathematical model
  - incompressible laminar viscous fluids flow
  - generalized Newtonian fluids model
  - stress tensor - Newtonian model
  - viscosity - Newtonian - $\mu = const.$
  - generalized Newtonian - power-law model

$$\mu = \mu(\dot{\gamma}) = \left(\sqrt{\mathrm{tr} D^2}\right)^r$$

- Numerical model
  - steady computation
  - finite volume method
  - artificial compressibility method
  - multistage Runge-Kutta method
  - unsteady computation
  - dual-time stepping method

Mathematical Model

The system of 2D Navier-Stokes equations in conservative form

$$\tilde{R}W_t + F^c_x + G^c_y = F^v_x + G^v_y \qquad \tilde{R} = \mathrm{diag}(0, 1, 1) \tag{1}$$

where

$$W = \begin{pmatrix} p \\ u \\ v \end{pmatrix} \quad F^c = \begin{pmatrix} u \\ u^2 + p \\ uv \end{pmatrix} \quad G^c = \begin{pmatrix} v \\ uv \\ v^2 + p \end{pmatrix}$$

$$F^v = \begin{pmatrix} 0 \\ t_{xx} \\ t_{yx} \end{pmatrix} \quad G^v = \begin{pmatrix} 0 \\ t_{xy} \\ t_{yy} \end{pmatrix}$$

Mathematical Model - Stress Tensor

Newtonian model

$$T = 2\mu D$$

$$T = \begin{pmatrix} t_{xx} & t_{xy} \\ t_{yx} & t_{yy} \end{pmatrix}$$

$$D_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$$

$$D = \frac{1}{2}\begin{pmatrix} 2u_x & u_y + v_x \\ u_y + v_x & 2v_y \end{pmatrix}$$

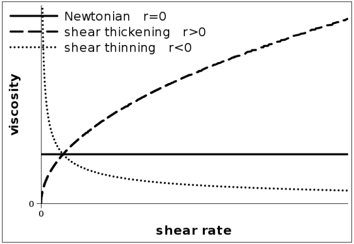
Mathematical Model - Generalized Newtonian Fluids

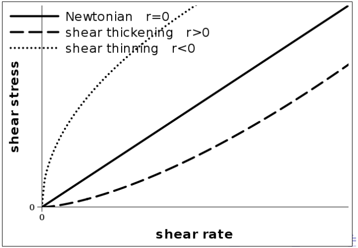
$$T = 2\mu_\epsilon \, \mu D$$

Power-law model

$$\mu = \mu(\dot{\gamma}) = \mu_\epsilon \left(\sqrt{trD^2}\right)^r$$

μ_ϵ - Newtonian viscosity
r - power-law index
$r = 0$ - Newtonian fluids flow
$r > 0$ - shear thickening fluids flow
$r < 0$ - shear thinning fluids flow





Steady Numerical Solution

Eq. (1) - steady state solution

$$\tilde{R}W_t + F^c_x + G^c_y = \frac{1}{\mathrm{Re}}(F^v_x + G^v_y), \quad \tilde{R}_\beta = \mathrm{diag}(0,1,1)$$

Artificial compressibility method

$$\tilde{R}_\beta W_t + F^c_x + G^c_y = \frac{1}{\mathrm{Re}}(F^v_x + G^v_y), \quad \tilde{R}_\beta = \mathrm{diag}(\frac{1}{\beta^2},1,1), \quad \mathrm{Re} = \frac{UL}{\mu_\epsilon}$$

$$W = \begin{pmatrix} p \\ u \\ v \end{pmatrix}, \quad F^c = \begin{pmatrix} u \\ u^2 + p \\ uv \end{pmatrix}, \quad G^c = \begin{pmatrix} v \\ uv \\ v^2 + p \end{pmatrix}$$

$$F^v = \begin{pmatrix} 0 \\ 2\mu(\dot{\gamma})u_x \\ \mu(\dot{\gamma})(v_x + u_y) \end{pmatrix}, \quad G^v = \begin{pmatrix} 0 \\ \mu(\dot{\gamma})(u_y + v_x) \\ 2\mu(\dot{\gamma})v_y \end{pmatrix}$$

Multistage Runge-Kutta Method

$$
\begin{aligned}
W_i^n &= W_i^{(0)} \\
W_i^{(s)} &= W_i^{(0)} - \alpha_s \Delta t \mathcal{R}(W)_i^{(s-1)} \\
W_i^{n+1} &= W_i^{(m)} \qquad\qquad s = 1, \dots, m
\end{aligned}
$$

$$
m = 3, \quad \alpha_1 = \alpha_2 = 0.5, \quad \alpha_3 = 1.
$$

steady residual

$$
\mathcal{R}(W)_i = \frac{1}{\mu_i} \sum_{k=1}^{4} \left[\left(\overline{F}_k^c - \frac{1}{\mathrm{Re}} \overline{F}_k^v \right) \Delta y_k - \left(\overline{G}_k^c - \frac{1}{\mathrm{Re}} \overline{G}_k^v \right) \Delta x_k \right]
$$

Numerical Fluxes

$\overline{F}^c_k, \overline{G}^c_k$ ... inviscid numerical fluxes

$$\overline{F}^c_k = F^c_k\left(\frac{W_i + W_k}{2}\right) \qquad \overline{G}^c_k = G^c_k\left(\frac{W_i + W_k}{2}\right)$$

dissipative numerical fluxes $\overline{F}^v_k, \overline{G}^v_k$ are computed using dual volume cells (Green formula)

numerical approximations of the velocity derivatives

$$\overline{u}_x = \frac{1}{\mu_k}\sum_{m=1}^{4} u_m \Delta y_m \qquad \overline{u}_y = -\frac{1}{\mu_k}\sum_{m=1}^{4} u_m \Delta x_m$$

Time Step and Convergence (Runge-Kutta Scheme)

The condition of stability at regular orthogonal grid

$$\Delta t = \min_{i,k} \frac{\mathrm{CFL}\mu_i}{\rho_A \Delta y_k + \rho_B \Delta x_k + \frac{2}{\mathrm{Re}}\mu(\dot{\gamma})\left(\frac{(\Delta x_k)^2+(\Delta y_k)^2}{\mu_i}\right)}$$

Convergence to steady state is checked by behavior of steady residual in the space L^2

$$\mathrm{Res}\,W^n = \sqrt{\frac{1}{num}\sum_i \left(\frac{W_i^{n+1} - W_i^n}{\Delta t}\right)^2}$$

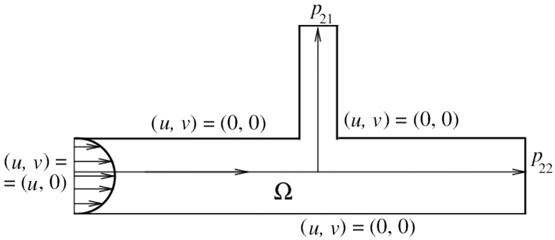
num - number of the finite volume cells
The decadic logarithm of $\|\mathrm{Res}(W)^n\|_{\mathrm{L}^2}$ is plotted in graphs presenting convergence history of simulation.

Boundary Conditions

Eq. (1) solved in the domain Ω

$$\tilde{R}W_t + F_x^c + G_y^c = F_x^v + G_y^v \qquad \tilde{R} = \mathrm{diag}(0,1,1)$$

- inlet conditions
  - Dirichlet BC
  parabolic profile
- outlet conditions
  - $p = p_{21}, p_{22}$,
- wall conditions
  - $(u,v)^T = (0,0)^T$
- other values of W
  - Neumann BC



Unsteady Numerical Solution

Dual-time stepping method

$$\tilde{R}_\beta W_\tau + \tilde{R} W_t + F^c_x + G^c_y = F^v_x + G^v_y$$

$$\tilde{R}_\beta = \mathrm{diag}(\frac{1}{\beta^2}, 1, 1) \quad \tilde{R} = \mathrm{diag}(0, 1, 1)$$

Three point backward formula

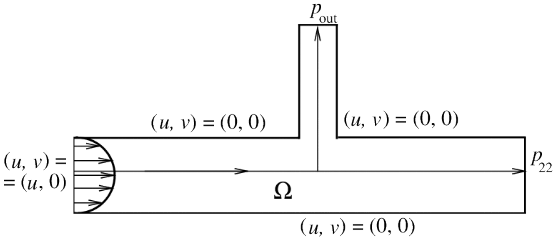
$$\tilde{R}_\beta \frac{W^{l+1} - W^l}{\Delta\tau} = -\tilde{R}\frac{3W^{l+1} - 4W^n + W^{n-1}}{2\Delta t} - \mathrm{Res}(W)^l = -\overline{\mathrm{Res}}(W)^{l+1}$$

l - dual time, n - physical time

$l \to \infty \Rightarrow n \to (n+1)$

Unsteady Boundary Condition

- inlet conditions
  - Dirichlet BC
  parabolic profile
- outlet conditions
  - $p = p_{22}$,
- wall conditions
  - $(u, v)^T = (0, 0)^T$
- other values of W
  - Neumann BC

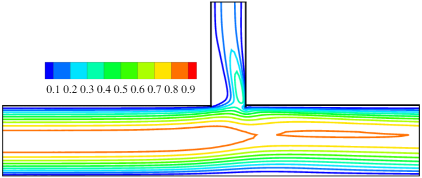


$$p_{out} = \frac{1}{4}\left(1 + \frac{1}{2}\sin(\omega t)\right)$$

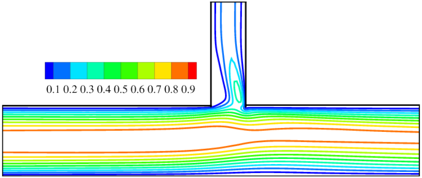
ω - angular velocity, $\omega = 2\pi f$
f - frequency

Steady Numerical Results - Re = 400

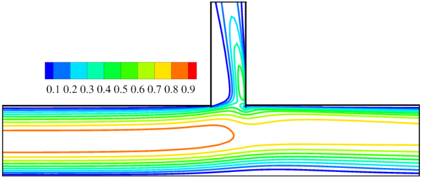
Newtonian fluids ($r = 0$)



Shear thickening Non-Newtonian fluids ($r = 0.5$)

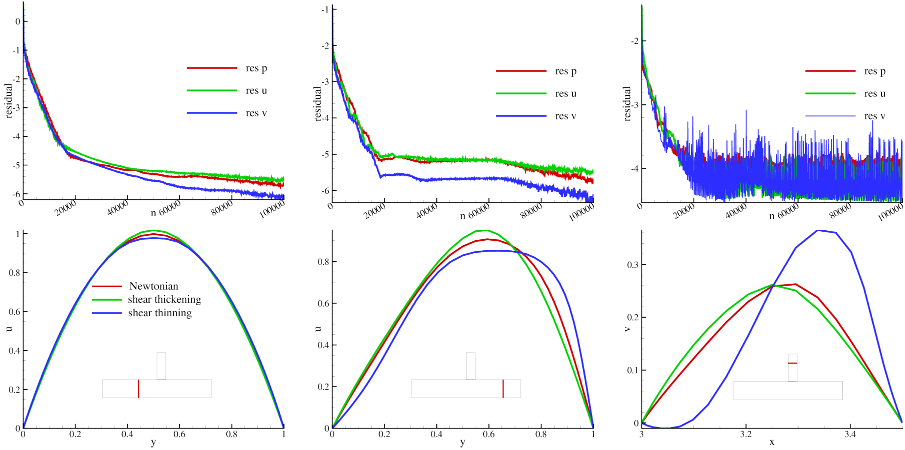


Shear thinning Non-Newtonian fluids ($r = -0.5$)



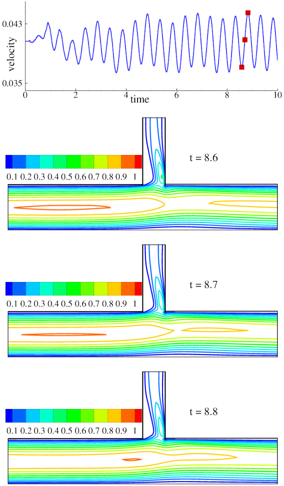
Steady Numerical Results

Newtonian fluids | Shear thickening fluids | Shear thinning fluids

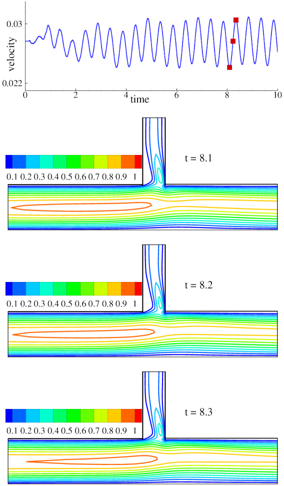


Unsteady Numerical Results, Artificial Compressibility Method, frequency f = 2

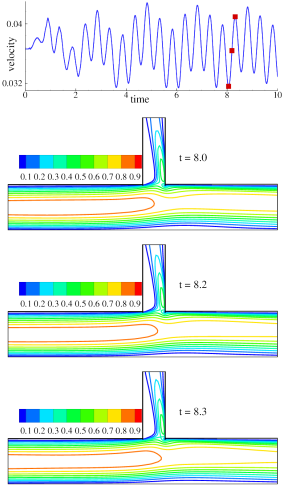
Newtonian



Shear thickening

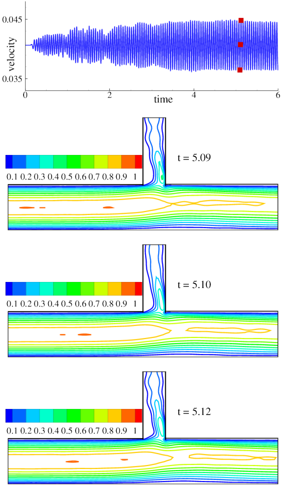


Shear thinning

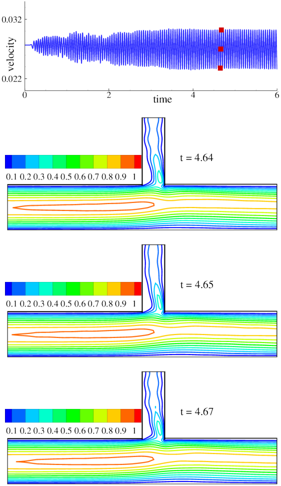


Unsteady Numerical Results, Artificial Compressibility Method, frequency f = 20

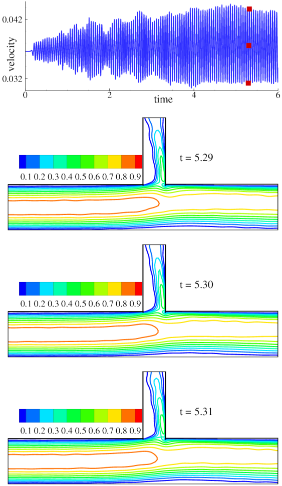
Newtonian



Shear thickening

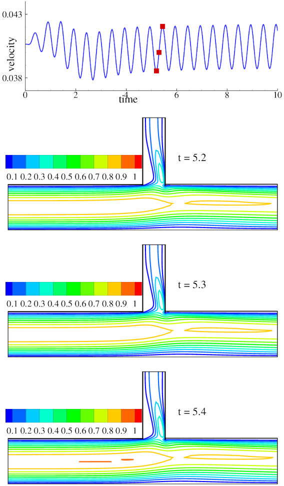


Shear thinning

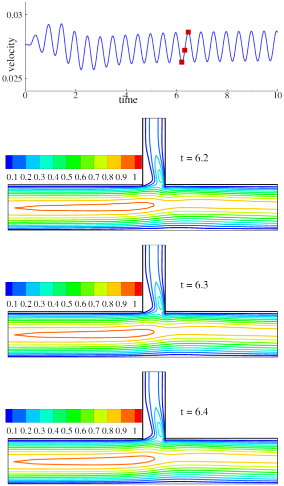


Unsteady Numerical Results, Dual-Time Stepping Method, frequency f = 2

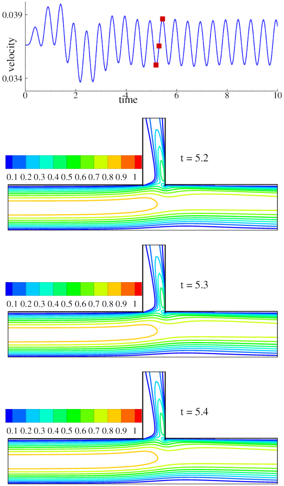
Newtonian



Shear thickening

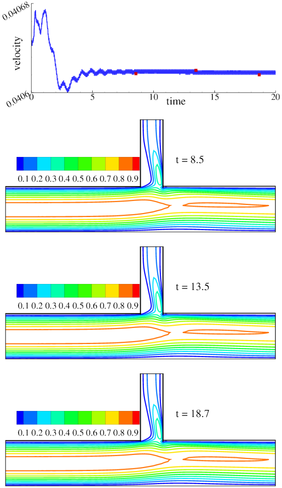


Shear thinning

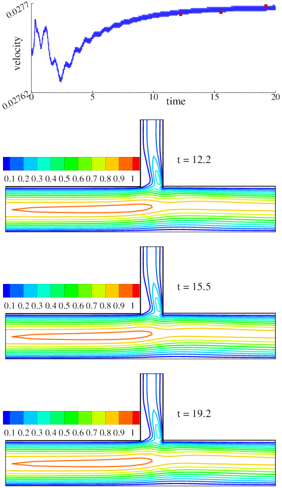


Unsteady Numerical Results, Dual-Time Stepping Method, frequency f = 20

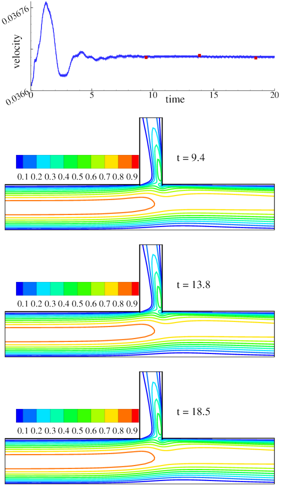
Newtonian



Shear thickening



Shear thinning



Conclusion

- Mathematical model for generalized Newtonian fluids (Newtonian and non-Newtonian shear thickening and shear thinning fluids)
- Numerical method for steady and unsteady simulations
- Finite volume method, Runge-Kutta scheme
- Unsteady computations - artificial compressibility method and dual time-stepping method